Problems and Solutions: CRMO-2012, Paper 4

1. Let $ABCD$ be a unit square. Draw a quadrant of a circle with $A$ as centre and $B, D$ as end points of the arc. Similarly, draw a quadrant of a circle with $B$ as centre and $A, C$ as end points of the arc. Inscribe a circle $\Gamma$ touching the arc $AC$ externally, the arc $BD$ internally and also touching the side $AD$. Find the radius of the circle $\Gamma$.

**Solution:** Let $O$ be the centre of $\Gamma$ and $r$ its radius. Draw $OP \perp AD$ and $OQ \perp AB$. Then $OP = r$, $OQ^2 = OA^2 - r^2 = (1 - r)^2 - r^2 = 1 - 2r$. We also have $OB = 1 + r$ and $BQ = 1 - r$. Using Pythagoras’ theorem we get

$$(1 + r)^2 = (1 - r)^2 + 1 - 2r.$$ Simplification gives $r = 1/6$.

2. Let $a, b, c$ be positive integers such that $a$ divides $b^2$, $b$ divides $c^2$ and $c$ divides $a^2$. Prove that $abc$ divides $(a + b + c)^7$.

**Solution:** If a prime $p$ divides $a$, then $p \mid b^2$ and hence $p \mid b$. This implies that $p \mid c^2$ and hence $p \mid c$. Thus every prime dividing $a$ also divides $b$ and $c$. By symmetry, this is true for $b$ and $c$ as well. We conclude that $a, b, c$ have the same set of prime divisors.

Let $p^x \mid a$, $p^y \mid b$ and $p^z \mid c$. (Here we write $p^x \mid a$ to mean $p^x$ divides $a$ and $p^{x+1} \mid a$.) We may assume $\min\{x, y, z\} = x$. Now $b \mid c^2$ implies that $y \leq 2z$; $c \mid a^2$ implies that $z \leq 2x$. We obtain

$$y \leq 2z \leq 4x.$$ 

Thus $x + y + z \leq x + 2x + 4x = 7x$. Hence the maximum power of $p$ that divides $abc$ is $x + y + z \leq 7x$. Since $x$ is the minimum among $x, y, z$, $p^x$ divides $a, b, c$. Hence $p^x$ divides $a + b + c$. This implies that $p^{7x}$ divides $(a + b + c)^7$. Since $x + y + z \leq 7x$, it follows that $p^{x+y+z}$ divides $(a + b + c)^7$. This is true of any prime $p$ dividing $a, b, c$. Hence $abc$ divides $(a + b + c)^7$.

3. Let $a$ and $b$ be positive real numbers such that $a + b = 1$. Prove that

$$a^a b^b + a^b b^a \leq 1.$$ 

**Solution:** Observe

$$1 = a + b = a^{a+b} b^{a+b} = a^a b^b + a^b b^a.$$ 

Hence

$$1 - a^a b^b - a^b b^a = a^a b^b + a^b b^a - a^a b^a - a^b b^a = (a^a - b^a)(b^b - b^b)$$

Now if $a \leq b$, then $a^a \leq b^a$ and $a^b \leq b^b$. If $a \geq b$, then $a^a \geq b^a$ and $a^b \geq b^b$. Hence the product is nonnegative for all positive $a$ and $b$. It follows that

$$a^a b^b + a^b b^a \leq 1.$$ 

4. Let $X = \{1, 2, 3, \ldots, 11\}$. Find the the number of pairs $\{A, B\}$ such that $A \subseteq X$, $B \subseteq X$, $A \neq B$ and $A \cap B = \{4, 5, 7, 8, 9, 10\}$.

**Solution:** Let $A \cup B = Y$, $B \setminus A = M$, $A \setminus B = N$ and $X \setminus Y = L$. Then $X$ is the disjoint union of $M$, $N$, $L$ and $A \cap B$. Now $A \cap B = \{4, 5, 7, 8, 9, 10\}$ is fixed. The remaining 5 elements 1, 2, 3, 6, 11 can be distributed in any of the remaining sets $M,$
N, L. This can be done in $3^5$ ways. Of these if all the elements are in the set L, then $A = B = \{4, 5, 7, 8, 9, 10\}$ and this case has to be omitted. Hence the total number of pairs \{A, B\} such that $A \subseteq X, B \subseteq X, A \neq B$ and $A \cap B = \{4, 5, 7, 8, 9, 10\}$ is $3^5 - 1$.

5. Let $ABC$ be a triangle. Let $E$ be a point on the segment $BC$ such that $BE = 2EC$. Let $F$ be the mid-point of $AC$. Let $BF$ intersect $AE$ in $Q$. Determine $BQ/QF$.

**Solution:** Let $CQ$ and $ET$ meet $AB$ in $S$ and $T$ respectively. We have

\[
\frac{[SBC]}{[ASC]} = \frac{BS}{SA} = \frac{[SBQ]}{[ASQ]}
\]

Using componendo by dividendo, we obtain

\[
\frac{BS}{SA} = \frac{[SBC] - [SBQ]}{[ASC] - [ASQ]} = \frac{[BQC]}{[AQC]}
\]

Similarly, We can prove

\[
\frac{BE}{EC} = \frac{[BQA]}{[CQA]'}
\]

\[
\frac{CF}{FA} = \frac{[CQB]}{[AQB]'}
\]

But $BD = DE = EC$ implies that $BE/EC = 2; CF = FA$ gives $CF/FA = 1$. Thus

\[
\frac{BS}{SA} = \frac{[BQC]}{[AQC]} = \frac{[BQC]/[AQB]}{[AQC]/[AQB]} = \frac{CF/FA}{EC/BE} = \frac{1}{1/2} = 2
\]

Now

\[
\frac{BQ}{QF} = \frac{[BQC] - [BQA]}{[FQC] - [FQA]} = \frac{[BQC]}{[FQC]} - \frac{[BQA]}{[FQA]} = \frac{[BQC]}{[AQC]} + \frac{[BQA]}{[AQC]} = \frac{[BQC] + [BQA]}{[AQC]}
\]

This gives

\[
\frac{BQ}{QF} = \frac{[BQC] + [BQA]}{[AQC]} = \frac{[BQC]}{[AQC]} + \frac{[BQA]}{[AQC]} = \frac{BS}{SA} + \frac{BE}{EC} = 2 + 2 = 4
\]

(Note: $BS/SA$ can also be obtained using Ceva’s theorem. One can also obtain the result by coordinate geometry.)

6. Solve the system of equations for positive real numbers:

\[
\frac{1}{xy} = \frac{1}{z} + 1, \quad \frac{1}{yz} = \frac{1}{x} + 1, \quad \frac{1}{zx} = \frac{z}{y} + 1
\]

**Solution:** The given system reduces to

\[
z = x^2y + xyz, \quad x = y^2 + xyz, \quad y = z^2x + xyz
\]

Hence

\[
z - x^2y = x - y^2z = y - z^2x
\]

If $x = y$, then $y^2z = z^2x$ and hence $x^2z = z^2x$. This implies that $z = x = y$. Similarly, $x = z$ implies that $x = z = y$. Hence if any two of $x, y, z$ are equal, then all are equal. Suppose no two of $x, y, z$ are equal. We may take $x$ is the largest among $x, y, z$ so that $x > y$ and $x > z$. Here we have two possibilities: $y > z$ and $z > y$.

Suppose $x > y > z$. Now $z - x^2y = x - y^2z = y - z^2x$ shows that

\[
y^2z > z^2x > x^2y
\]
But \( y^2z > z^2x \) and \( z^2x > x^2y \) give \( y^2 > z^2 \) and \( z^2 > xy \). Hence

\[
(y^2)(z^2) > (zx)(xy).
\]

This gives \( yz > x^2 \). Thus \( x^3 < xyz = (xz)y < (y^2)y = y^3 \). This forces \( x < y \) contradicting \( x > y \).

Similarly, we arrive at a contradiction if \( x > z > y \). The only possibility is \( x = y = z \).

For \( x = y = z \), we get only one equation \( x^2 = 1/2 \). Since \( x > 0 \), \( x = 1/\sqrt{2} = y = z \).

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