1. Let $ABC$ be a triangle and $D$ be a point on the segment $BC$ such that $DC = 2BD$. Let $E$ be the mid-point of $AC$. Let $AD$ and $BE$ intersect in $P$. Determine the ratios $BP/PE$ and $AP/PD$.

**Solution:** Let $F$ be the midpoint of $DC$, so that $D, F$ are points of trisection of $BC$. Now in triangle $CAD$, $F$ is the mid-point of $CD$ and $E$ is that of $CA$. Hence $CF/FD = 1 = CE/EA$. Thus $EF \parallel AD$. Hence we find that $EF \parallel PD$. Hence $BP/PE = BD/DF$. But $BD = DF$. We obtain $BP/PE = 1$.

In triangle $ACD$, since $EF \parallel AD$ we get $EF/AD = CF/CD = 1/2$. Thus $AD = 2EF$. But $PD/EF = BD/BF = 1/2$. Hence $EF = 2PD$. Therefore this gives

$$AP = AD - PD = 3PD.$$ We obtain $AP/PD = 3$.

(Coordinate geometry proof is also possible.)

2. Let $a, b, c$ be positive integers such that $a$ divides $b^3$, $b$ divides $c^3$ and $c$ divides $a^3$. Prove that $abc$ divides $(a + b + c)^3$.

**Solution:** If a prime $p$ divides $a$, then $p | b^3$ and hence $p | b$. This implies that $p | c^3$ and hence $p | c$. Thus every prime dividing $a$ also divides $b$ and $c$. By symmetry, this is true for $b$ and $c$ as well. We conclude that $a, b, c$ have the same set of prime divisors.

Let $p^x || a$, $p^y || b$ and $p^z || c$. (Here we write $p^x || a$ to mean $p^x | a$ and $p^{x+1} \nmid a$.) We may assume $\min \{x, y, z\} = x$. Now $b | c^3$ implies that $y \leq 3z$; $c | a^3$ implies that $z \leq 3x$. We obtain

$$y \leq 3z \leq 9x.$$Thus $x + y + z \leq x + 3x + 9x = 13x$. Hence the maximum power of $p$ that divides $abc$ is $x + y + z \leq 13x$. Since $x$ is the minimum among $x, y, z$, whence $p^x$ divides each of $a, b, c$. Hence $p^x$ divides $a + b + c$. This implies that $p^{13x}$ divides $(a + b + c)^3$. Since $x + y + z \leq 13x$, it follows that $p^{x+y+z}$ divides $(a + b + c)^3$. This is true of any prime $p$ dividing $a, b, c$. Hence $abc$ divides $(a + b + c)^3$.

3. Let $a$ and $b$ be positive real numbers such that $a + b = 1$. Prove that

$$a^a b^b + a^b b^a \leq 1.$$**Solution:** Observe

$$1 = a + b = a^{a+b} b^{a+b} = a^a b^b + b^a a^b.$$Hence

$$1 - a^a b^b - a^b b^a = a^a b^b + b^a a^b - a^a b^b - a^b b^a = (a^a - b^a)(a^b - b^b)$$Now if $a \leq b$, then $a^a \leq b^a$ and $a^b \leq b^b$. If $a > b$, then $a^a \geq b^a$ and $a^b \geq b^b$. Hence the product is nonnegative for all positive $a$ and $b$. It follows that

$$a^a b^b + a^b b^a \leq 1.$$
4. Let \( X = \{1, 2, 3, \ldots, 10\} \). Find the number of pairs \( \{A, B\} \) such that \( A \subseteq X, B \subseteq X, A \neq B \) and \( A \cap B = \{2, 3, 5, 7\} \).

**Solution:** Let \( A \cup B = Y, B \setminus A = M, A \setminus B = N \) and \( X \setminus Y = L \). Then \( X \) is the disjoint union of \( M, N, L \) and \( A \cap B \). Now \( A \cap B = \{2, 3, 5, 7\} \) is fixed. The remaining six elements \( 1, 4, 6, 8, 9, 10 \) can be distributed in any of the remaining sets \( M, N, L \). This can be done in \( 3^6 \) ways. Of these if all the elements are in the set \( L \), then \( A = B = \{2, 3, 5, 7\} \) and which this case has to be deleted. Hence the total number of pairs \( \{A, B\} \) such that \( A \subseteq X, B \subseteq X, A \neq B \) and \( A \cap B = \{2, 3, 5, 7\} \) is \( 3^6 - 1 \).

5. Let \( ABC \) be a triangle. Let \( BE \) and \( CF \) be internal angle bisectors of \( \angle B \) and \( \angle C \) respectively with \( E \) on \( AC \) and \( F \) on \( AB \). Suppose \( X \) is a point on the segment \( CF \) such that \( AX \perp CF \); and \( Y \) is a point on the segment \( BE \) such that \( AY \perp BE \). Prove that \( XY = (b + c - a)/2 \) where \( BC = a, CA = b \) and \( AB = c \).

**Solution:** Produce \( AX \) and \( AY \) to meet \( BC \) is \( X' \) and \( Y' \) respectively. Since \( BY \) bisects \( \angle ABY' \) and \( AY' \perp AB \) it follows that \( BA = BY' \) and \( AY = YY' \). Similarly, \( CA = CX' \) and \( AX = XX' \). Thus \( X \) and \( Y \) are mid-points of \( AY' \) and \( AX' \) respectively. By mid-point theorem \( XY = X'Y'/2 \). But

\[
X'Y' = X'C + Y'B - BC = AC + AB - BC = b + c - a.
\]

Hence \( XY = (b + c - a)/2 \).

6. Let \( a \) and \( b \) be real numbers such that \( a \neq 0 \). Prove that not all the roots of \( ax^4 + bx^3 + x^2 + x + 1 = 0 \) can be real.

**Solution:** Let \( \alpha_1, \alpha_2, \alpha_3, \alpha_4 \) be the roots of \( ax^4 + bx^3 + x^2 + x + 1 = 0 \). Observe none of these is zero since their product is \( 1/a \). Then the roots of \( x^4 + x^3 + x^2 + bx + a = 0 \) are

\[
\beta_1 = \frac{1}{\alpha_1}, \beta_2 = \frac{1}{\alpha_2}, \beta_3 = \frac{1}{\alpha_3}, \beta_4 = \frac{1}{\alpha_4}.
\]

We have

\[
\sum_{j=1}^{4} \beta_j = -1, \quad \sum_{1 \leq j < k \leq 4} 4\beta_j \beta_k = 1.
\]

Hence

\[
\sum_{j=1}^{4} \beta_j^2 = \left( \sum_{j=1}^{4} \beta_j \right)^2 - 2 \left( \sum_{1 \leq j < k \leq 4} \beta_j \beta_k \right) = 1 - 2 = -1.
\]

This shows that not all \( \beta_j \) can be real. Hence not all \( \alpha_j \)'s can be real.