1. Prove that there do not exist natural numbers \(x\) and \(y\), with \(x > 1\), such that
\[
\frac{x^7 - 1}{x - 1} = y^5 + 1.
\]

**Solution.** Simple factorisation gives
\[
y^5 = x(x^3 + 1)(x^2 + x + 1).
\]
The three factors on the right are mutually coprime and hence they all have to be fifth powers. In particular, \(x = r^5\) for some integer \(r\). This implies \(x^3 + 1 = r^{15} + 1\), which is not a fifth power unless \(r = -1\) or \(r = 0\). This implies there are no solutions to the given equation.

2. In a triangle \(ABC\), \(AD\) is the altitude from \(A\), and \(H\) is the orthocentre. Let \(K\) be the centre of the circle passing through \(D\) and tangent to \(BH\) at \(H\). Prove that the line \(DK\) bisects \(AC\).

**Solution.** Note that \(\angle KHB = 90^\circ\). Therefore \(\angle KDA = \angle KHD = 90^\circ - \angle BHD = \angle HBD = \angle HAC\). On the other hand, if \(M\) is the midpoint of \(AC\) then it is the circumcenter of triangle \(ADC\) and therefore \(\angle MDA = \angle MAD\). This proves that \(D, K, M\) are collinear and hence \(DK\) bisects \(AC\).

3. Consider the expression
\[
2013^2 + 2014^2 + 2015^2 + \cdots + n^2.
\]
Prove that there exists a natural number \(n > 2013\) for which one can change a suitable number of plus signs to minus signs in the above expression to make the resulting expression equal 9999.

**Solution.** For any integer \(k\) we have
\[
-k^2 + (k + 1)^2 + (k + 2)^2 - (k + 3)^2 = -4.
\]
Note that \(9999 - (2013^2 + 2014^2 + 2015^2 + 2016^2 + 2017^2) = -4m\) for some positive integer \(m\). Therefore, it follows that
\[
9999 = (2013^2 + 2014^2 + 2015^2 + 2016^2 + 2017^2)
+ \sum_{r=1}^{m} \left(- (4r + 2014)^2 + (4r + 2015)^2 + (4r + 2016)^2 - (4r + 2017)^2\right).
\]

4. Let \(ABC\) be a triangle with \(\angle A = 90^\circ\) and \(AB = AC\). Let \(D\) and \(E\) be points on the segment \(BC\) such that \(BD : DE : EC = 1 : 2 : \sqrt{3}\). Prove that \(\angle DAE = 45^\circ\).

**Solution.** Rotating the configuration about \(A\) by \(90^\circ\), the point \(B\) goes to the point \(C\). Let \(P\) denote the image of the point \(D\) under this rotation. Then \(CP = BD\) and \(\angle ACP = \angle ABC = 45^\circ\), so \(ECP\) is a right-angled triangle with \(CE : CP = \sqrt{3} : 1\). Hence \(PE = ED\). It follows that \(ADEP\) is a kite with \(AP = AD\) and \(PE = ED\). Therefore \(AE\) is the angular bisector of \(\angle PAD\). This implies that \(\angle DAE = \angle PAD/2 = 45^\circ\).
5. Let \( n \geq 3 \) be a natural number and let \( P \) be a polygon with \( n \) sides. Let \( a_1, a_2, \ldots, a_n \) be the lengths of the sides of \( P \) and let \( p \) be its perimeter. Prove that
\[
\frac{a_1}{p-a_1} + \frac{a_2}{p-a_2} + \cdots + \frac{a_n}{p-a_n} < 2.
\]

**Solution.** If \( r \) and \( s \) are positive real numbers such that \( r < s \) then \( r/s < (r+x)/(s+x) \) for any positive real \( x \). Note that, by triangle inequality, \( a_i < p - a_i \), so
\[
\frac{a_i}{p-a_i} < \frac{2a_i}{p},
\]
for all \( i = 1, 2, \ldots, n \). Summing this inequality over \( i \) we get the desired inequality.

6. For a natural number \( n \), let \( T(n) \) denote the number of ways we can place \( n \) objects of weights \( 1, 2, \ldots, n \) on a balance such that the sum of the weights in each pan is the same. Prove that \( T(100) > T(99) \).

**Solution.** Let \( S(n) \) denote the collection of subsets \( A \) of \( X(n) = \{1, 2, \ldots, n\} \) such that the sum of the elements of \( A \) equals \( n(n+1)/4 \). Then the given inequality is equivalent to \(|S(100)| > |S(99)|\). We shall give a map \( f : S(99) \to S(100) \) which is one-to-one but not onto. Note that this will prove the required inequality.

Suppose that \( A \) is an element of \( S(99) \). If \( 50 \in A \) then define \( f(A) = (A \setminus \{50\}) \cup \{100\} \). Otherwise, define \( f(A) = A \cup \{50\} \). If \( A \) and \( B \) are elements of \( S(99) \) such that \( f(A) = f(B) \) then either 50 belongs to both these sets or neither of these sets. In either of the cases we have \( A = B \). Therefore \( f \) is a one-to-one function.

Note that every element in the range of \( f \) contains exactly one of 50 and 100. Let \( B_i = \{i, 101 - i\} \). Then \( B_1 \cup B_2 \cup \cdots B_{24} \cup B_{50} \) is an element of \( S(100) \). Clearly, this is not in the range of \( f \). Thus \( f \) is not an onto function.