Regional Mathematical Olympiad-2015

Time: 3 hours

December 06, 2015

Instructions:

• Calculators (in any form) and protractors are not allowed.
• Rulers and compasses are allowed.
• Answer all the questions.
• All questions carry equal marks. Maximum marks: 102.
• Answer to each question should start on a new page. Clearly indicate the question number.

1. Let \(ABC\) be a triangle. Let \(B'\) denote the reflection of \(B\) in the internal angle bisector \(\ell\) of \(\angle A\). Show that the circumcentre of the triangle \(CB'I\) lies on the line \(\ell\), where \(I\) is the incentre of \(ABC\).

2. Let \(P(x) = x^2 + ax + b\) be a quadratic polynomial where \(a\) is real and \(b\) is rational. Suppose \(P(0)^2, P(1)^2, P(2)^2\) are integers. Prove that \(a\) and \(b\) are integers.

3. Find all integers \(a, b, c\) such that

\[
a^2 = bc + 4, \quad b^2 = ca + 4.
\]

4. Suppose 40 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

5. Two circles \(\Gamma\) and \(\Sigma\) intersect at two distinct points \(A\) and \(B\). A line through \(B\) intersects \(\Gamma\) and \(\Sigma\) again at \(C\) and \(D\), respectively. Suppose that \(CA = CD\). Show that the centre of \(\Sigma\) lies on \(\Gamma\).

6. How many integers \(m\) satisfy both the following properties:

(i) \(1 \leq m \leq 5000\); (ii) \([\sqrt{m}] = [\sqrt{m + 125}]\)?

(Here \([x]\) denotes the largest integer not exceeding \(x\), for any real number \(x\).)