Regional Mathematical Olympiad-2015

Time: 3 hours December 06, 2015

Instructions:

• Calculators (in any form) and protractors are not allowed.

• Rulers and compasses are allowed.

• Answer all the questions.

• All questions carry equal marks. Maximum marks: 102.

• Answer to each question should start on a new page. Clearly indicate the question number.

1. Two circles $\Gamma$ and $\Sigma$, with centres $O$ and $O'$, respectively, are such that $O'$ lies on $\Gamma$. Let $A$ be a point on $\Sigma$ and $M$ the midpoint of the segment $AO'$. If $B$ is a point on $\Sigma$ different from $A$ such that $AB$ is parallel to $OM$, show that the midpoint of $AB$ lies on $\Gamma$.

2. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial where $a$ and $b$ are real numbers. Suppose $(P(-1)^2, P(0)^2, P(1)^2)$ is an arithmetic progression of integers. Prove that $a$ and $b$ are integers.

3. Show that there are infinitely many triples $(x, y, z)$ of integers such that $x^3 + y^4 = z^{31}$.

4. Suppose 36 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

5. Let $ABC$ be a triangle with circumcircle $\Gamma$ and incentre $I$. Let the internal angle bisectors of $\angle A$, $\angle B$ and $\angle C$ meet $\Gamma$ in $A'$, $B'$ and $C'$ respectively. Let $B'C'$ intersect $AA'$ in $P$ and $AC$ in $Q$, and let $BB'$ intersect $AC$ in $R$. Suppose the quadrilateral $PIRQ$ is a kite; that is, $IP = IR$ and $QP = QR$. Prove that $ABC$ is an equilateral triangle.

6. Show that there are infinitely many positive real numbers $a$ which are not integers such that $a(a - 3\{a\})$ is an integer. (Here $\{a\}$ denotes the fractional part of $a$. For example $\{1.5\} = 0.5$; $\{-3.4\} = 0.6$.)