Regional Mathematical Olympiad-2015

Time: 3 hours

December 06, 2015

Instructions:

• Calculators (in any form) and protractors are not allowed.
• Rulers and compasses are allowed.
• Answer all the questions.
• All questions carry equal marks. Maximum marks: 102.
• Answer to each question should start on a new page. Clearly indicate the question number.

1. Let \( \triangle ABC \) be a triangle. Let \( B' \) and \( C' \) denote respectively the reflection of \( B \) and \( C \) in the internal angle bisector of \( \angle A \). Show that the triangles \( \triangle ABC \) and \( \triangle AB'C' \) have the same incentre.

2. Let \( P(x) = x^2 + ax + b \) be a quadratic polynomial with real coefficients. Suppose there are real numbers \( s \neq t \) such that \( P(s) = t \) and \( P(t) = s \). Prove that \( b - st \) is a root of the equation \( x^2 + ax + b - st = 0 \).

3. Find all integers \( a, b, c \) such that
   \[
a^2 = bc + 1, \quad b^2 = ca + 1.
   \]

4. Suppose 32 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

5. Two circles \( \Gamma \) and \( \Sigma \) in the plane intersect at two distinct points \( A \) and \( B \), and the centre of \( \Sigma \) lies on \( \Gamma \). Let points \( C \) and \( D \) be on \( \Gamma \) and \( \Sigma \), respectively, such that \( C, B \) and \( D \) are collinear. Let point \( E \) on \( \Sigma \) be such that \( DE \) is parallel to \( AC \). Show that \( AE = AB \).

6. Find all real numbers \( a \) such that \( 4 < a < 5 \) and \( a(a - 3\{a\}) \) is an integer. (Here \( \{a\} \) denotes the fractional part of \( a \). For example \( \{1.5\} = 0.5; \{-3.4\} = 0.6 \).)