Regional Mathematical Olympiad 2014 (Mumbai region)

- There are six questions in this question paper. Answer all questions.
- Each question carries 10 points.
- Use of protractors, calculators, mobile phone is forbidden.
- Time allotted: 3 hours

1. Three positive real numbers $a$, $b$, $c$ are such that $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = 0$. Can $a$, $b$, $c$ be the lengths of the sides of a triangle? Justify your answer.

2. The roots of the equation $x^3 - 3ax^2 + bx + 18c = 0$ form a non-constant arithmetic progression and the roots of the equation $x^3 + bx^2 + x - c^3 = 0$ form a non-constant geometric progression. Given that $a$, $b$, $c$ are real numbers, find all positive integral values $a$ and $b$.

3. Let $ABC$ be an acute-angled triangle in which $\angle ABC$ is the largest angle. Let $O$ be its circumcentre. The perpendicular bisectors of $BC$ and $AB$ meet $AC$ at $X$ and $Y$ respectively. The internal bisectors of $\angle AXB$ and $\angle BYC$ meet $AB$ and $BC$ at $D$ and $E$ respectively. Prove that $BO$ is perpendicular to $AC$ if $DE$ is parallel to $AC$.

4. A person moves in the $x - y$ plane moving along points with integer co-ordinates $x$ and $y$ only. When she is at point $(x, y)$, she takes a step based on the following rules:

(a) if $x + y$ is even she moves to either $(x + 1, y)$ or $(x + 1, y + 1)$;
(b) if $x + y$ is odd she moves to either $(x, y + 1)$ or $(x + 1, y + 1)$.

How many distinct paths can she take to go from $(0, 0)$ to $(8, 8)$ given that she took exactly three steps to the right ($(x, y)$ to $(x + 1, y)$)?

5. Let $a$, $b$, $c$ be positive numbers such that

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1.$$ 

Prove that $(1 + a^2)(1 + b^2)(1 + c^2) \geq 125$. When does the equality hold?

6. Let $D$, $E$, $F$ be the points of contact of the incircle of an acute-angled triangle $ABC$ with $BC$, $CA$, $AB$ respectively. Let $I_1$, $I_2$, $I_3$ be the incentres of the triangles $AFE$, $BDF$, $CED$, respectively. Prove that the lines $I_1D$, $I_2E$, $I_3F$ are concurrent.

GOOD LUCK