Regional Mathematical Olympiad-2014

Time: 3 hours December 07, 2014

Instructions:

• Calculators (in any form) and protractors are not allowed.
• Rulers and compasses are allowed.
• Answer all the questions.
• All questions carry equal marks. Maximum marks: 102.
• Answer to each question should start on a new page. Clearly indicate the question number.

1. Let \(ABC\) be a triangle and let \(AD\) be the perpendicular from \(A\) on to \(BC\). Let \(K, L, M\) be points on \(AD\) such that \(AK = KL = LM = MD\). If the sum of the areas of the shaded regions is equal to the sum of the areas of the unshaded regions, prove that \(BD = DC\).

2. Let \(a_1, a_2, \ldots, a_{2n}\) be an arithmetic progression of positive real numbers with common difference \(d\). Let

(i) \(a_1^2 + a_3^2 + \cdots + a_{2n-1}^2 = x\),

(ii) \(a_2^2 + a_4^2 + \cdots + a_{2n}^2 = y\), and

(iii) \(a_n + a_{n+1} = z\).

Express \(d\) in terms of \(x, y, z, n\).

3. Suppose for some positive integers \(r\) and \(s\), the digits of \(2^r\) is obtained by permuting the digits of \(2^s\) in decimal expansion. Prove that \(r = s\).

4. Is it possible to write the numbers 17, 18, 19, \ldots, 32 in a \(4 \times 4\) grid of unit squares, with one number in each square, such that the product of the numbers in each \(2 \times 2\) sub-grids \(AMRG, GRND, MBHR\) and \(RHCN\) is divisible by 16?

5. Let \(ABC\) be an acute-angled triangle and let \(H\) be its ortho-centre. For any point \(P\) on the circum-circle of triangle \(ABC\), let \(Q\) be the point of intersection of the line \(BH\) with the line \(AP\). Show that there is a unique point \(X\) on the circum-circle of \(ABC\) such that for every point \(P \neq A, B\), the circum-circle of \(HQP\) pass through \(X\).

6. Let \(x_1, x_2, \ldots, x_{2014}\) be positive real numbers such that \(\sum_{j=1}^{2014} x_j = 1\). Determine with proof the smallest constant \(K\) such that

\[K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1.\]