Regional Mathematical Olympiad-2018

Solutions

1. Let ABC be a triangle with integer sides in which $AB < AC$. Let the tangent to the circumcircle of triangle ABC at A intersect the line BC at D. Suppose AD is also an integer. Prove that $\gcd(AB, AC) > 1$.

   Solution: We may assume that B lies between C and D. Let $AB = c, BC = a$ and $CA = b$. Then $b > c$. Let $BD = x$ and $AD = y$. Observe thast $\angle DAB = \angle DCA$. Hence $\triangle DAB \sim \triangle DCA$. We get

   $$\frac{x}{y} = \frac{c}{b} = \frac{y}{x+a}.$$

   Therefore $xb = yc$ and $by = c(x + a)$. Eliminating x, we get $y = abc/(b^2 - c^2)$.

   Suppose $\gcd(b, c) = 1$. Then $\gcd(b, b^2 - c^2) = 1 = \gcd(c, b^2 - c^2)$. Since y is an integer, $b^2 - c^2$ divides a. Therefore $b + c$ divides a. Hence

   $$a \geq b + c.$$

   This contradicts triangle inequality. We conclude that $\gcd(b, c) > 1$.

2. Let n be a natural number. Find all real numbers x satsfying the equation

   $$\sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}} = \frac{n(n+1)}{4}.$$

   Solution: Observe that $x \neq 0$. We also have

   $$\begin{aligned} \frac{n(n+1)}{4} = \left|\sum_{k=1}^{n} \frac{kx^k}{1+x^{2k}}\right| &\leq \sum_{k=1}^{n} \frac{k|x|^k}{1+x^{2k}}. \\ &= \sum_{k=1}^{n} \frac{k}{\frac{1}{|x|^k} + |x|^k} \\ &\leq \sum_{k=1}^{n} \frac{k}{2} = \frac{n(n+1)}{4}. \end{aligned}$$

   Hence equality holds every where. It follows that $x = |x|$ and $|x| = 1/|x|$. We conclude that $x = 1$ is the unique solution to the equation.

3. For a rational number r, its *period* is the length of the smallest repeating block in its decimal expansion. For example, the number $r = 0.123123123\cdots$ has period 3. If S denotes the set of all rational numbers r of the form $r = 0.\overline{abcdefgh}$ having period 8, find the sum of all the elements of S.

   Solution: Let us first count the number of elements in S. There are 10^8 ways of choosing a block of length 8. Of these, we shoud not count the blocks of the form *abcdabcd*, *abababab*, and *aaaaaaaa*. There are 10^4 blocks of the form *abcdabcd*. They include blocks of the form *abababab* and *aaaaaaaa*. Hence the blocks of length exactly 8 is $10^8 - 10^4 = 99990000$.

For each block $abcdefgh$ consider the block $a'b'c'd'e'f'g'h'$ where $x' = 9 - x$. Observe that whenever $0.\overline{abcdefgh}$ is in S, the rational number $0.\overline{a'b'c'd'e'f'g'h'}$ is also in S. Thus every element $0.\overline{abcdefgh}$ of S can be uniquely paired with a distinct element $0.\overline{a'b'c'd'e'f'g'h'}$ of S. We also observe that

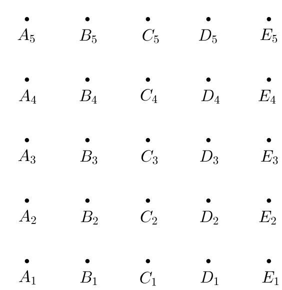
$$0.\overline{abcdefgh} + 0.\overline{a'b'c'd'e'f'g'h'} = 0.\overline{99999999} = 1.$$

Hence the sum of elements in S is

$$\frac{99990000}{2} = 49995000.$$

4. Let E denote the set of 25 points (m, n) in the xy-plane, where m, n are natural numbers, $1 \leq m \leq 5$, $1 \leq n \leq 5$. Suppose the points of E are arbitrarily coloured using two colours, red and blue. Show that there always exist four points in the set E of the form (a, b), $(a+k, b)$, $(a+k, b+k)$, $(a, b+k)$ for some positive integer k such that at least three of these four points have the same colour. (That is, there always exist four points in the set E which form the vertices of a square and having at least three points of the same colour.)

Solution: Name the points from bottom row to top (and from left to right) as A_j, B_j, C_j, D_j, E_j, $1 \leq j \leq 5$.



Note that among 5 points A_1, B_1, C_1, D_1, E_1, there are at least 3 points of the same colour, say, *red*. (This folllows from pigeonhole principle.) We consider several cases: (the argument holds irrespective of the colour assigned to the other two points.)
(I) Take three adjacent points having the same colour. (e.g. A_1, B_1, C_1 or B_1, C_1, D_1.) The argument is similar in both the cases. If A_1, B_1, C_1 are red then A_2, B_2, C_2 are all blue; otherwise we get a square having three red vertices. The same reasoning shows that A_3, B_3, C_3 are all red. Now A_1, C_1, A_3, C_3 have all red vertices.

(II) Three alternate points A_1, C_1, E_1 which are red: Then A_3, C_3, E_3 have to be blue; otherwise, we get a square with three red vertices. Same reasoning shows that A_5, C_5, E_5 are red. Therefore we have A_1, E_1, A_5, E_5 have red colour.
(III) Only two adjacent points having red colour: There are three sub cases.
(a) A_1, B_1, D_1 red: In this case A_2, B_2 are blue and therefore A_3, B_3 are red. But then B_1, D_1, B_3 are red vertices of a square.
(b) B_1, C_1, E_1 are red. This is similar to case (a).
(c) A_1, B_1, E_1 are red. We successively have A_2, B_2 blue; A_3, B_3 red; A_4, B_4 blue; A_5, B_5 red. Now A_1, E_1, A_5 are the red vertices of a square.
These are the only essential cases and all other reduce to one of these cases.

5. Find all natural numbers n such that $1 + [\sqrt{2n}]$ divides $2n$. (For any real number x, $[x]$ denotes the largest integer not exceeding x.)

Solution: Let $[\sqrt{2n}] = k$. We observe that $x - 1 < [x] \leq x$. Hence

$$\sqrt{2n} < 1 + k \leq 1 + \sqrt{2n}.$$

Divisibility gives $(1+k)d = 2n$ for some positive integer d. Therefore we obtain

$$\sqrt{2n} < \frac{2n}{d} \leq 1 + \sqrt{2n}.$$

The first inequality gives $d < \sqrt{2n} < 1 + k$. But then

$$d = \frac{2n}{1+k} = \frac{(\sqrt{2n})^2}{1+k} \geq \frac{k^2}{1+k} = (k-1) + \frac{1}{k+1} > k - 1.$$

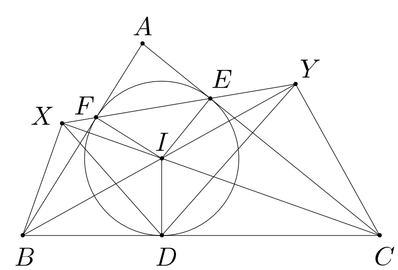
We thus obtain $k - 1 < d < k + 1$. Since d is an integer, it follows that $d = k$. This implies that $n = k(k+1)/2$. Thus n is a triangular number. It is easy to check that every triangular number is a solution.

6. Let ABC be an acute-angled triangle with $AB < AC$. Let I be the incentre of triangle ABC, and let D, E, F be the points at which its incircle touches the sides BC, CA, AB, respectively. Let BI, CI meet the line EF at Y, X, respectively. Further assume that both X and Y are outside the triangle ABC. Prove that
(i) B, C, Y, X are concyclic; and
(ii) I is also the incentre of triangle DYX.

Solution:
(a) We first show that $BIFX$ is a cyclic quadrilateral. Since $\angle BIC = 90° + (A/2)$, we see that $\angle BIX = 90° - (A/2)$. On the otherhand FAE is an isosceles triangle so that $\angle AFE = 90° - (A/2)$. But $\angle AFE = \angle BFX$ as they are vertically opposite angles. Therefore $\angle BFX = 90° - (A/2) = \angle BIX$. It follows that $BIFX$ is a cyclic quadrilateral. Therefore $\angle BXI = \angle BFI$. But $\angle BFI = 90°$ since $IF \perp AB$. We obtain $\angle BXC = \angle BXI = 90°$.

A similar consideration shows that $\angle BYC = 90°$. Therefore $\angle BXC = \angle BYC$ which implies that $BCYX$ is a cyclic quadrilateral.

(b) We also observe that $BDIX$ is a cyclic quadrilateral as $\angle BXI = 90° = \angle BDI$ and therefore $\angle BXI + \angle BDI = 180°$. This gives $\angle DXI = \angle DBI = B/2$. Now the concyclicity of B, I, F, X shows that $\angle IXF = \angle IBF = B/2$. Hence $\angle DXI = \angle IXF$. Hence XI bisects $\angle DXY$. Similarly, we can show that YI bisects $\angle DYX$. It follows that I is the incentre of $\triangle DYX$ as well.



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