Regional Mathematical Olympiad-2016

Time: 3 hours **October 16, 2016**

Instructions:

- **Calculators (in any form) and protractors are not allowed.**
- **Rulers and compasses are allowed.**
- **Answer all the questions.**
- **All questions carry equal marks. Maximum marks: 102.**
- **Answer to each question should start on a new page. Clearly indicate the question number.**

1. Let ABC be a triangle and D be the mid-point of BC. Suppose the angle bisector of $\angle ADC$ is tangent to the circumcircle of triangle ABD at D. Prove that $\angle A = 90^\circ$.

2. Let a, b, c be three distinct positive real numbers such that $abc = 1$. Prove that
$$\frac{a^3}{(a-b)(a-c)} + \frac{b^3}{(b-c)(b-a)} + \frac{c^3}{(c-a)(c-b)} \geq 3.$$

3. Let a, b, c, d, e, f be positive integers such that
$$\frac{a}{b} < \frac{c}{d} < \frac{e}{f}.$$
Suppose $af - be = -1$. Show that $d \geq b + f$.

4. There are 100 countries participating in an olympiad. Suppose n is a positive integer such that each of the 100 countries is willing to communicate in exactly n languages. If each set of 20 countries can communicate in at least one common language, and no language is common to all 100 countries, what is the minimum possible value of n?

5. Let ABC be a right-angled triangle with $\angle B = 90^\circ$. Let I be the incentre of ABC. Extend AI and CI; let them intersect BC in D and AB in E respectively. Draw a line perpendicular to AI at I to meet AC in J; draw a line perpendicular to CI at I to meet AC in K. Suppose $DJ = EK$. Prove that $BA = BC$.

6. (a) Given any natural number N, prove that there exists a strictly increasing sequence of N positive integers in harmonic progression.

   (b) Prove that there cannot exist a strictly increasing infinite sequence of positive integers which is in harmonic progression.

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