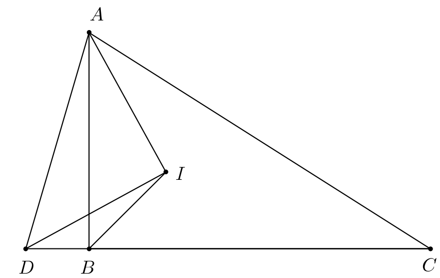
1. Let ABC be a right-angled triangle with $\angle B = 90°$. Let I be the incentre of ABC. Draw a line perpendicular to AI at I. Let it intersect the line CB at D. Prove that CI is perpendicular to AD and prove that $ID = \sqrt{b(b-a)}$ where $BC = a$ and $CA = b$.

   Solution: First observe that $ADBI$ is a cyclic quadrilateral since $\angle AID = \angle ABD = 90°$. Hence $\angle ADI = \angle ABI = 45°$. Hence $\angle DAI = 45°$. But we also have



$$\angle ADB = \angle ADI + \angle IDB = 45° + \angle IAB$$
$$= \angle DAI + \angle IAC = \angle DAC.$$

   Therefore CDA is an isosceles triangle with $CD = CA$. Since CI bisects $\angle C$ it follows that $CI \perp AD$.

   This shows that $DB = CA - CB = b - a$. Therefore

$$AD^2 = c^2 + (b-a)^2 = c^2 + b^2 + a^2 - 2ba = 2b(b-a).$$

   But then $2ID^2 = AD^2 = 2b(b-a)$ and this gives $ID = \sqrt{b(b-a)}$.

2. Let a, b, c be positive real numbers such that

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} = 1.$$

   Prove that $abc \leq 1/8$.

   Solution: This is equivalent to

$$\sum a(1+b)(1+c) = (1+a)(1+b)(1+c).$$

   This simplifies to

$$ab + bc + ca + 2abc = 1$$

   Using AM-GM inequality, we have

$$1 = ab + bc + ca + 2abc \geq 4(ab \cdot bc \cdot ca \cdot 2abc)^{1/4}.$$

   Simplificaton gives

$$abc \leq \frac{1}{8}.$$

3. For any natural number n, expressed in base 10, let $S(n)$ denote the sum of all digits of n. Find all natural numbers n such that $n = 2S(n)^2$.

   Solution: We use the fact that 9 divides $n - S(n)$ for every natural number n. Hence $S(n)(2S(n)-1)$ is divisible by 9. Since $S(n)$ and $2S(n) - 1$ are relatively prime, it follows that 9 divides either $S(n)$ or $2S(n) - 1$, but not both. We also observe that the number of digits of n cannot exceed 4. If n has k digits, then $n \geq 10^{k-1}$ and $2S(n)^2 \leq 2 \times (9k)^2 = 162k^2$. If $k \geq 6$, we see that

$$2S(n)^2 \leq 162k^2 < 5^4k^2 < 10^{k-1} \leq n.$$

   If $k = 5$, we have

$$2S(n)^2 \leq 162 \times 25 = 4150 < 10^4 \leq n.$$

Therefore $n \leq 4$ and $S(n) \leq 36$.
If $9|S(n)$, then $S(n) = 9, 18, 27, 36$. We see that $2S(n)^2$ is respectively equal to 162, 648, 1458, 2592. Only 162 and 648 satisfy $n = 2S(n)^2$.
If $9|(2S(n)-1)$, then $2S(n) = 9k+1$. Only $k = 1, 3, 5, 7$ give integer values for $S(n)$. In these cases $2S(n)^2 = 50, 392, 1058, 2048$. Here again 50 and 392 give $n = 2S(n)^2$.
Thus the only natural numbers wth the property $n = 2S(n)^2$ are $50, 162, 392, 648$.

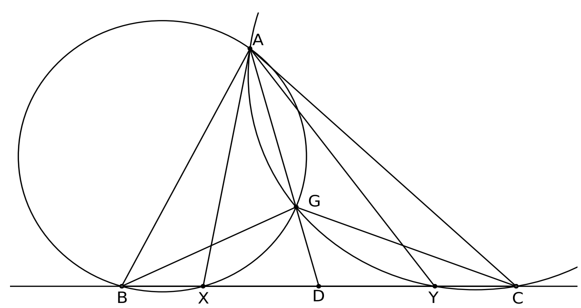
4. Find the number of all 6-digit natural numbers having exactly three odd digits and three even digits.

   Solution: First we choose 3 places for even digits. This can be done in $\binom{6}{3} = 20$ ways. Observe that the other places for odd digits get automatically fixed. There are 5 even digits and 5 odd digits. Any of these can occur in their proper places. Hence there are 5^6 ways of selecting 3 even and 3 odd digits for a particular selection of place for even digits. Hence we get 20×5^6 such numbers. But this includes all those numbers having the first digit equal to 0. Since we are looking for 6-digit numbers, these numbers have to be removed from our counting. If we fix 0 as the first digit, we have, 2 places for even numbers and 3 places for odd numbers. We can choose 2 places for even numbers in $\binom{5}{2} = 10$ ways. As earlier, for any such choice of places for even digits, we can choose even digits in 5^2 ways and odd digits in 5^3 ways. Hence the number of ways of choosing 3 even and 3 odd digits with 0 as the first digit is 10×5^5. Therfore the number of 6-digit numbers with 3 even digits and 3 odd digits is

$$20 \times 5^6 - 10 \times 5^5 = 10 \times 5^5(10-1) = 281250.$$

5. Let ABC be a triangle with centroid G. Let the circumcircles of $\triangle AGB$ and $\triangle AGC$ intersect the line BC in X and Y respectively, which are distinct from B, C. Prove that G is the centroid of $\triangle AXY$.

   Solution: Let D be the midpoint of AB. Observe that $DX \cdot DB = DG \cdot DA = DY \cdot DC$. But $DB = DC$. Hence $DX = DY$. This means that D is the midpoint of XY as well. Hence AD is also a median of $\triangle AXY$. Now we know that $AG : GD = 2 : 1$. If G' is the median of $\triangle AXY$, then G' must lie on AD and $AG' : G'D = 2 : 1$. We conclude that $G = G'$.



6. Let $\langle a_1, a_2, a_3, \ldots \rangle$ be a strictly increasing sequence of positive integers in an arithmetic progression. Prove that there is an infinite subsequence of the given sequence whose terms are in a geometric progression.

   Solution: Let $\langle a_1, a_2, \ldots, a_{n+1} \ldots \rangle = \langle a, a+d, \ldots, a+nd, \ldots \rangle$ be a strictly increasing sequence of positive integers in arithmetic progression. Here a and d are both positive integers. Consider the following subsequence:

$$\langle a, a(1+d), a(1+d)^2, \ldots, a(1+d)^n, \ldots \rangle.$$

   This is a geometric progression. Here $a > 0$ and the common ratio $1+d > 1$. Hence the sequence is strictly increasing. The first term is a which is in the given AP. The second term is $a(1+d) = a+ad$ which is the $(a+1)$-th term of the AP. In general, we see that

$$a(1+d)^n = a + d\left(\binom{n}{1}a + \binom{n}{2}ad + \cdots + \binom{n}{n}ad^{n-1}\right).$$

Here the coefficient of d in the braces is also a positive integer. Hence $a(1+d)^n$ is also a term of the given AP.