1. Let ABC be an isosceles triangle with $AB = AC$ and let Γ denote its circumcircle. A point D is on the arc AB of Γ not containing C and a point E is on the arc AC of Γ not containing B such that $AD = CE$. Prove that BE is parallel to AD.

   Solution. We note that triangle AEC and triangle BDA are congruent. Therefore $AE = BD$ and hence $\angle ABE = \angle DAB$. This proves that AD is parallel to BE. □

2. Find all triples (p, q, r) of primes such that $pq = r + 1$ and $2(p^2 + q^2) = r^2 + 1$.

   Solution. If p and q are both odd, then $r = pq - 1$ is even so $r = 2$. But in this case $pq \geq 3 \times 3 = 9$ and hence there are no solutions. This proves that either $p = 2$ or $q = 2$. If $p = 2$ then we have $2q = r + 1$ and $8 + 2q^2 = r^2 + 1$. Multiplying the second equation by 2 we get $2r^2 + 2 = 16 + (2q)^2 = 16 + (r + 1)^2$. Rearranging the terms, we have $r^2 - 2r - 15 = 0$, or equivalently $(r + 3)(r - 5) = 0$. This proves that $r = 5$ and hence $q = 3$. Similarly, if $q = 2$ then $r = 5$ and $p = 3$. Thus the only two solutions are $(p, q, r) = (2, 3, 5)$ and $(p, q, r) = (3, 2, 5)$. □

3. A finite non-empty set S of integers is called 3-*good* if the the sum of the elements of S is divisble by 3. Find the number of 3-good non-empty subsets of $\{0, 1, 2, \ldots, 9\}$.

   Solution. Let A be a 3-good subset of $\{0, 1, \ldots, 9\}$. Let $A_1 = A \cap \{0, 3, 6, 9\}$, $A_2 = A \cap \{1, 4, 7\}$ and $A_3 = A \cap \{2, 5, 8\}$. Then there are three possibilities:

   - $|A_2| = 3, |A_3| = 0$;
   - $|A_2| = 0, |A_3| = 3$;
   - $|A_2| = |A_3|$.

   Note that there are 16 possibilities for A_1. Therefore the first two cases correspond to a total of 32 subsets that are 3-good. The number of subsets in the last case is $16(1^2+3^2+3^2+1^2) = 320$. Note that this also includes the empty set. Therefore there are a total of 351 non-empty 3-good subsets of $\{0, 1, 2, \ldots, 9\}$. □

4. In a triangle ABC, points D and E are on segments BC and AC such that $BD = 3DC$ and $AE = 4EC$. Point P is on line ED such that D is the midpoint of segment EP. Lines AP and BC intersect at point S. Find the ratio BS/SD.

   Solution. Let F denote the midpoint of the segment AE. Then it follows that DF is parallel to AP. Therefore, in triangle ASC we have $CD/SD = CF/FA = 3/2$. But $DC = BD/3 = (BS + SD)/3$. Therefore $BS/SD = 7/2$. □

5. Let a_1, b_1, c_1 be natural numbers. We define

   $$a_2 = gcd(b_1, c_1)\,, \quad b_2 = gcd(c_1, a_1)\,, \quad c_2 = gcd(a_1, b_1)\,,$$

   and

   $$a_3 = lcm(b_2, c_2)\,, \quad b_3 = lcm(c_2, a_2)\,, \quad c_3 = lcm(a_2, b_2)\,.$$

   Show that $gcd(b_3, c_3) = a_2$.

Solution. For a prime p and a natural number n we shall denote by $v_p(n)$ the power of p dividing n. Then it is enough to show that $v_p(a_2) = v_p(gcd(b_3, c_3))$ for all primes p. Let p be a prime and let $\alpha = v_p(a_1), \beta = v_p(b_1)$ and $\gamma = v_p(c_1)$. Because of symmetry, we may assume that $\alpha \leq \beta \leq \gamma$. Therefore, $v_p(a_2) = \min\{\beta, \gamma\} = \beta$ and similarly $v_p(b_2) = v_p(c_2) = \alpha$. Therefore $v_p(b_3) = \max\{\alpha, \beta\} = \beta$ and similarly $v_p(c_3) = \max\{\alpha, \beta\} = \beta$. Therefore $v_p(gcd(b_3, c_3)) = v_p(a_2) = \beta$. This completes the solution. □

6. Let a, b be real numbers and, let $P(x) = x^3 + ax^2 + b$ and $Q(x) = x^3 + bx + a$. Suppose that the roots of the equation $P(x) = 0$ are the reciprocals of the roots of the equation $Q(x) = 0$. Find the greatest common divisor of $P(2013! + 1)$ and $Q(2013! + 1)$.

Solution. Note that $P(0) \neq 0$. Let $R(x) = x^3P(1/x) = bx^3 + ax + 1$. Then the equations $Q(x) = 0$ and $R(x) = 0$ have the same roots. This implies that $R(x) = bQ(x)$ and equating the coefficients we get $a = b^2$ and $ab = 1$. This implies that $b^3 = 1$, so $a = b = 1$. Thus $P(x) = x^3 + x^2 + 1$ and $Q(x) = x^3 + x + 1$. For any integer n we have

$$(P(n), Q(n)) = (P(n), P(n) - Q(n)) = (n^3 + n^2 + 1, n^2 - n) = (n^3 + n^2 + 1, n - 1) = (3, n - 1)\,.$$

Thus $(P(n), Q(n)) = 3$ if $n - 1$ is divisible by 3. In particular, since 3 divides 2013! it follows that $(P(2013! + 1), Q(2013! + 1)) = 3$.

□