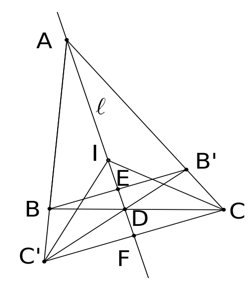
CRMO-2015 questions and solutions

1. Let ABC be a triangle. Let B' and C' denote respectively the reflection of B and C in the internal angle bisector of $\angle A$. Show that the triangles ABC and $AB'C'$ have the same incentre.

   Solution: Join BB' and CC'. Let the internal angle bisector ℓ of $\angle A$ meet BB' in E and CC' in F. Since B' is the reflection of B in ℓ, we observe that $BB' \perp \ell$ and $BE = EB'$. Hence B' lies on AC. Similarly, C' lies on the line AB.

   Let D be the point of intersection of BC and $B'C'$. Observe that $BB' \parallel C'C$. Moreover the triangles ABC is congruent to $AB'C'$: this follows from the observation that $AB = AB'$ and $AC = AC'$ and the included angle $\angle A$ is common. Hence $BC' = B'C$ so that $C'CB'B$ is an isosceles trapezium. This means that the intesection point D of its diagonal lies on the perpendicular bisector of its parallel sides. Thus ℓ passes through D. We also observe that $CD = C'D$.
   Let I be the incentre of $\triangle ABC$. This means that CI bisects $\angle C$. Hence $AI/ID = AC/CD$. But $AC = AC'$ and $CD = C'D$. Hence we also get that $AI/ID = AC'/C'D$. This implies that $C'I$ bisects $\angle AC'B'$. Therefore the two angle bisectors of $\triangle AC'B'$ meet at I. This shows that I is also the incentre of $\triangle AC'B'$.

2. Let $P(x) = x^2 + ax + b$ be a quadratic polynomial with real coefficients. Suppose there are real numbers $s \neq t$ such that $P(s) = t$ and $P(t) = s$. Prove that $b - st$ is a root of the equation $x^2 + ax + b - st = 0$.

   Solution: We have

$$\begin{aligned} s^2 + as + b &= t, \\ t^2 + at + b &= s. \end{aligned}$$

   This gives

$$(s^2 - t^2) + a(s - t) = (t - s).$$

   Since $s \neq t$, we obtain $s + t + a = -1$. Adding the equations, we obtain

$$s^2 + t^2 + a(s + t) + 2b = (s + t).$$

   Therefore

$$(s + t)^2 - 2st + a(s + t) + 2b = (s + t).$$

   Using $s + t = -(1 + a)$, we obtain

$$(1 + a)^2 - 2st - a(1 + a) + 2b = -1 - a.$$

   Simplification gives $st = 1+a+b = P(1)$. This shows that $x = 1$ is a root of $x^2+ax+b-st = 0$. Since the product of roots is $b - st$, the other root is $b - st$.

3. Find all integers a, b, c such that

$$a^2 = bc + 1, \quad b^2 = ca + 1.$$

Solution: Suppose $a = b$. Then we get one equation: $a^2 = ac + 1$. This reduces to $a(a - c) = 1$. Therefore $a = 1$, $a - c = 1$; and $a = -1$, $a - c = -1$. Thus we get $(a, b, c) = (1, 1, 0)$ and $(-1, -1, 0)$.

If $a \neq b$, subtracting the second relation from the first we get

$$a^2 - b^2 = c(b - a).$$

This gives $a + b = -c$. Substituting this in the first equation, we get

$$a^2 = b(-a - b) + 1.$$

Thus $a^2 + b^2 + ab = 1$. Multiplication by 2 gives

$$(a + b)^2 + a^2 + b^2 = 2.$$

Thus $(a, b) = (1, -1), (-1, 1), (1, 0), (-1, 0), (0, 1), (0, -1)$. We get respectively $c = 0, 0, -1, 1, -1, 1$. Thus we get the triples:

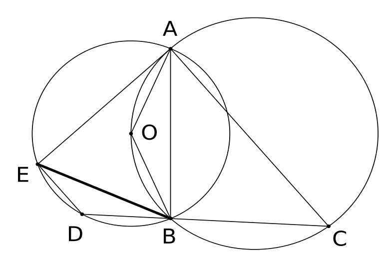
$$(a, b, c) = (1, 1, 0), (-1, -1, 0), (1, -1, 0), (-1, 1, 0), (1, 0, -1), (-1, 0, 1), (0, 1, -1), (0, -1, 1).$$

4. Suppose 32 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Solution: One can choose 3 objects out of 32 objects in $\binom{32}{3}$ ways. Among these choices all would be together in 32 cases; exactly two will be together in 32×28 cases. Thus three objects can be chosen such that no two adjacent in $\binom{32}{3} - 32 - (32 \times 28)$ ways. Among these, furthrer, two objects will be diametrically opposite in 16 ways and the third would be on either semicircle in a non adjacent portion in $32 - 6 = 26$ ways. Thus required number is

$$\binom{32}{3} - 32 - (32 \times 28) - (16 \times 26) = 3616.$$

5. Two circles Γ and Σ in the plane intersect at two distinct points A and B, and the centre of Σ lies on Γ. Let points C and D be on Γ and Σ, respectively, such that C, B and D are collinear. Let point E on Σ be such that DE is parallel to AC. Show that $AE = AB$.



Solution: If O is the centre of Σ, then we have

$$\angle AEB = \frac{1}{2}\angle AOB = \frac{1}{2}(180^\circ - \angle ACB)$$
$$= \frac{1}{2}\angle EDB = \frac{1}{2}\big(180^\circ - \angle EAB\big) = 90^\circ - \frac{1}{2}\angle EAB\,.$$

But we know that $\angle AEB + \angle EAB + \angle EBA = 180^\circ$.

Therefore

$$\angle EBA = 180^\circ - \angle AEB - \angle EAB = 180^\circ - 90^\circ + \frac{1}{2}\angle EAB - \angle EAB = 90^\circ - \frac{1}{2}\angle EAB.$$

This shows that $\angle AEB = \angle EBA$ and hence $AE = AB$.

6. Find all real numbers a such that $4 < a < 5$ and $a(a-3\{a\})$ is an integer. (Here $\{a\}$ denotes the fractional part of a. For example $\{1.5\} = 0.5$; $\{-3.4\} = 0.6$.)

Solution: Let $a = 4 + f$, where $0 < f < 1$. We are given that $(4+f)(4-2f)$ is an integer. This implies that $2f^2 + 4f$ is an integer. Since $0 < f < 1$, we have $0 < 2f^2 + 4f < 6$. Therefore $2f^2 + 4f$ can take $1, 2, 3, 4$ or 5. Equating $2f^2 + 4f$ to each one of them and using $f > 0$, we get

$$f = \frac{-2+\sqrt{6}}{2}, \frac{-2+\sqrt{8}}{2}, \frac{-2+\sqrt{10}}{2}, \frac{-2+\sqrt{12}}{2}, \frac{-2+\sqrt{14}}{2}.$$

Therefore a takes the values:

$$a = 3 + \frac{\sqrt{6}}{2}, 3 + \frac{\sqrt{8}}{2}, 3 + \frac{\sqrt{10}}{2}, 3 + \frac{\sqrt{12}}{2}, 3 + \frac{\sqrt{14}}{2}.$$

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