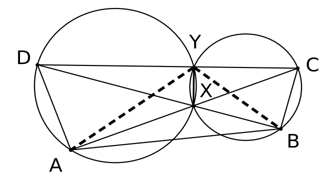
CRMO-2015 questions and solutions

1. In a cyclic quadrilateral $ABCD$, let the diagonals AC and BD intersect at X. Let the circumcircles of triangles AXD and BXC intersect again at Y. If X is the incentre of triangle ABY, show that $\angle CAD = 90°$.

   Solution: Given that X is the incentre of triangle ABY, we have $\angle BAX = \angle XAY$. Therefore, $\angle BDC = \angle BAC = \angle BAX = \angle XAY = \angle XDY = \angle BDY$. This shows that C, D, Y are collinear. Therefore, $\angle CYX + \angle XYD = 180°$. But the left-hand side equals $(180° - \angle CBD) + (180° - \angle CAD)$. Since $\angle CBD = \angle CAD$, we obtain $180° = 360° - 2\angle CAD$. This shows that $\angle CAD = 90°$.

2. Let $P_1(x) = x^2 + a_1x + b_1$ and $P_2(x) = x^2 + a_2x + b_2$ be two quadratic polynomials with integer coefficients. Suppose $a_1 \neq a_2$ and there exist integers $m \neq n$ such that $P_1(m) = P_2(n)$, $P_2(m) = P_1(n)$. Prove that $a_1 - a_2$ is even.

   Solution: We have

$$\begin{aligned} m^2 + a_1m + b_1 &= n^2 + a_2n + b_2 \\ n^2 + a_1n + b_1 &= m^2 + a_2m + b_2. \end{aligned}$$

   Hence

$$(a_1 - a_2)(m + n) = 2(b_2 - b_1), \quad (a_1 + a_2)(m - n) = 2(n^2 - m^2).$$

   This shows that $a_1 + a_2 = -2(n + m)$. Hence

$$4(b_2 - b_1) = a_2^2 - a_1^2.$$

   Since $a_1 + a_2$ and $a_1 - a_2$ have same parity, it follows that $a_1 - a_2$ is even.

3. Find all fractions which can be written simultaneously in the forms $\frac{7k-5}{5k-3}$ and $\frac{6l-1}{4l-3}$, for some integers k, l.

   Solution: If a fraction is simultaneously in the forms $\frac{7k-5}{5k-3}$ and $\frac{6l-1}{4l-3}$, we must have

$$\frac{7k-5}{5k-3} = \frac{6l-1}{4l-3}.$$

   This simplifies to $kl + 8k + l - 6 = 0$. We can write this in the form

$$(k + 1)(l + 8) = 14.$$

   Now 14 can be factored in 8 ways: 1×14, 2×7, 7×2, 14×1, $(-1) \times (-14)$, $(-2) \times (-7)$, $(-7) \times (-2)$ and $(-14) \times (-1)$. Thus we get 8 pairs:

$$(k, l) = (13, -7), (6, -6), (1, -1), (0, 6), (-15, -9), (-8, -10), (-3, -15), (-2, -22).$$

   These lead respectively to 8 fractions:

$$\frac{43}{31}, \quad \frac{31}{27}, \quad 1, \quad \frac{55}{39}, \quad \frac{5}{3}, \quad \frac{61}{43}, \quad \frac{19}{13}, \quad \frac{13}{9}.$$

4. Suppose 28 objects are placed along a circle at equal distances. In how many ways can 3 objects be chosen from among them so that no two of the three chosen objects are adjacent nor diametrically opposite?

Solution: One can choose 3 objects out of 28 objects in $\binom{28}{3}$ ways. Among these choices all would be together in 28 cases; exactly two will be together in 28×24 cases. Thus three objects can be chosen such that no two adjacent in $\binom{28}{3} - 28 - (28 \times 24)$ ways. Among these, furthrer, two objects will be diametrically opposite in 14 ways and the third would be on either semicircle in a non adjacent portion in $28 - 6 = 22$ ways. Thus required number is

$$\binom{28}{3} - 28 - (28 \times 24) - (14 \times 22) = 2268.$$

5. Let ABC be a right triangle with $\angle B = 90°$. Let E and F be respectively the mid-points of AB and AC. Suppose the incentre I of triangle ABC lies on the circumcircle of triangle AEF. Find the ratio BC/AB.

Solution: Draw $ID \perp AC$. Then $ID = r$, the inradius of $\triangle ABC$. Observe $EF \parallel BC$ and hence $\angle AEF = \angle ABC = 90°$. Hence $\angle AIF = 90°$. Therefore $ID^2 = FD \cdot DA$. If $a > c$, then $FA > DA$ and we have

$$DA = s - a, \quad \text{and} \quad FD = FA - DA = \frac{b}{2} - (s - a).$$

Thus we obtain

$$r^2 = \frac{(b + c - a)(a - c)}{4}.$$

But $r = (c + a - b)/2$. Thus we obtain

$$(c + a - b)^2 = (b + c - a)(a - c).$$

Simplification gives $3b = 3a + c$. Squaring both sides and using $b^2 = c^2 + a^2$, we obtian $4c = 3a$. Hence $BC/BA = a/c = 4/3$.

(If $a \leq c$, then I lies outside the circumcircle of AEF.)

6. Find all real numbers a such that $3 < a < 4$ and $a(a - 3\{a\})$ is an integer. (Here $\{a\}$ denotes the fractional part of a. For example $\{1.5\} = 0.5$; $\{-3.4\} = 0.6$.)

Solution: Let $a = 3 + f$, where $0 < f < 1$. We are given that $(3 + f)(3 - 2f)$ is an integer. This implies that $2f^2 + 3f$ is an integer. Since $0 < f < 1$, we have $0 < 2f^2 + 3f < 5$. Therefore $2f^2 + 3f$ can take $1, 2, 3$ or 4. Equating $2f^2 + 3f$ to each one of them and using $f > 0$, we get

$$f = \frac{-3 + \sqrt{17}}{4}, \quad \frac{1}{2}, \quad \frac{-3 + \sqrt{33}}{4}, \quad \frac{-3 + \sqrt{41}}{4}.$$

Therefore a takes the values:

$$a = 3 + \frac{-3 + \sqrt{17}}{4}, \quad 3\frac{1}{2}, \quad 3 + \frac{-3 + \sqrt{33}}{4}, \quad 3 + \frac{-3 + \sqrt{41}}{4}.$$

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