Solutions to problems of RMO 2014 (Mumbai region)

1. **Three positive real numbers a, b, c are such that $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = 0$. Can a, b, c be the lengths of the sides of a triangle? Justify your answer.**

Solution

No. Note that $a^2 + 5b^2 + 4c^2 - 4ab - 4bc = (a - 2b)^2 + (b - 2c)^2 = 0 \Rightarrow a : b : c = 4 : 2 : 1 \Rightarrow b + c : a = 3 : 4$. The triangle inequality is violated.

2. **The roots of the equation**

$$x^3 - 3ax^2 + bx + 18c = 0$$

form a non-constant arithmetic progression and the roots of the equation

$$x^3 + bx^2 + x - c^3 = 0$$

form a non-constant geometric progression. Given that a, b, c are real numbers, find all positive integral values a and b.

Solution

Let $\alpha - d$, α, $\alpha + d$ ($d \neq 0$) be the roots of the first equation and let β/r, β, βr ($r > 0$ and $r \neq 1$) be the roots of the second equation. It follows that $\alpha = a$, $\beta = c$ and

$$a^3 - ad^2 = -18c; \quad 3a^2 - d^2 = b, \tag{1}$$

$$c(1/r + 1 + r) = -b; \quad c^2(1/r + 1 + r) = 1. \tag{2}$$

Eliminating d, r and c yields

$$ab^2 - 2a^3b - 18 = 0, \tag{3}$$

whence $b = a^2 \pm (1/a)\sqrt{a^6 + 18a}$. For positive integral values of a and b it must be that $a^6 + 18a$ is a perfect square. Let $x^2 = a^6 + 18a$. Then $a^3 < x^2 < a^3 + 1$ for $a > 2$ and hence no solution. For $a = 1$ there is no solution. For $a = 2$, $x = 10$ and $b = 9$. Thus the admissible pair is $(a, b) = (2, 9)$.

3. **Let ABC be an acute-angled triangle in which $\angle ABC$ is the largest angle. Let O be its circumcentre. The perpendicular bisectors of BC and AB meet AC at X and Y respectively. The internal bisectors of $\angle AXB$ and $\angle BYC$ meet AB and BC at D and E respectively. Prove that BO is perpendicular to AC if DE is parallel to AC.**

Solution

Observe that triangles AYB and BXC are isosceles ($AY = BY$ and $BX = CX$). This implies $\angle BYC = 2\angle BAC$ and $\angle AXB = 2\angle ACB$. Since XD and YE are angle bisectors we have $\angle AXD = \angle ACB$ and $\angle CYE = \angle CAB$. Hence XD is parallel to BC and YE is parallel to AB. Therefore

$$\frac{CE}{EB} = \frac{CY}{AY} \tag{4}$$

and

$$\frac{AD}{DB} = \frac{AX}{CX}. \tag{5}$$

Now, if DE is parallel to AC then $\dfrac{CE}{EB} = \dfrac{AD}{DB}$. Therefore we must have

$$\frac{CY}{AY} = \frac{AX}{CX}. \tag{6}$$

But then

$$\frac{CY}{AY} + 1 = \frac{AX}{CX} + 1 \Rightarrow \frac{AC}{AY} = \frac{AC}{CX} \Rightarrow AY = CX. \tag{7}$$

Hence $BY = AY = CX = BX$. Thus $\angle BXY = \angle BYX$ i.e $\angle AXB = \angle BYC$ or $\angle ACB = \angle BAC$ i.e triangle ABC is isosceles with $AB = CB$. Hence BO is the perpendicular bisector of AC.

4. **A person moves in the $x - y$ plane moving along points with integer co-ordinates x and y only. When she is at point (x, y), she takes a step based on the following rules:**

   (a) if $x + y$ is even she moves to either $(x + 1, y)$ or $(x + 1, y + 1)$;

   (b) if $x + y$ is odd she moves to either $(x, y + 1)$ or $(x + 1, y + 1)$.

   How many distinct paths can she take to go from $(0, 0)$ to $(8, 8)$ given that she took exactly three steps to the right $((x, y)$ to $(x + 1, y))$?

Solution

We note that she must also take three up steps and five diagonal steps. Now, a step to the right or an upstep changes the parity of the co-ordinate sum, and a diagonal step does not change it. Therefore, between two right steps there must be an upstep and similarly between two upsteps there must be a right step. We may, therefore write

$$HVHVHV$$

The diagonal steps may be distributed in any fashion before, in between and after the HV sequence. The required number is nothing but the number of ways of distributing 5 identical objects into 7 distinct boxes and is equal to $\binom{11}{6}$.

5. **Let a, b, c be positive numbers such that**

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1.$$

   Prove that $(1 + a^2)(1 + b^2)(1 + c^2) \geq 125$. When does the equality hold?

Solution

$$\frac{1}{1+a} + \frac{1}{1+b} + \frac{1}{1+c} \leq 1 \Rightarrow \frac{a}{1+a} \geq \frac{1}{1+b} + \frac{1}{1+c}. \tag{8}$$

Similarly,

$$\frac{b}{1+b} \geq \frac{1}{1+a} + \frac{1}{1+c}, \quad \frac{c}{1+c} \geq \frac{1}{1+a} + \frac{1}{1+c}. \tag{9}$$

Apply AM-GM to get that

$$\frac{a}{1+a} \geq \frac{2}{\sqrt{(1+b)(1+c)}}, \quad \frac{b}{1+b} \geq \frac{2}{\sqrt{(1+a)(1+c)}}, \quad \frac{c}{1+c} \geq \frac{2}{\sqrt{(1+a)(1+b)}}. \tag{10}$$

Multiplying these results we get

$$abc \geq 8. \tag{11}$$

Now take

$$F = (1+a^2)(1+b^2)(1+c^2) \geq 1 + a^2 + b^2 + c^2 + a^2b^2 + b^2c^2 + c^2a^2 + a^2b^2c^2 \tag{12}$$

and apply AM-GM to a^2, b^2, c^2 and to a^2b^2, b^2c^2, c^2a^2 to get

$$F \geq 1 + 3(a^2b^2c^2)^{1/3} + 3(a^4b^4c^4)^{1/3} + a^2b^2c^2 = [1 + (a^2b^2c^2)^{1/3}]^3 \geq [1 + 8^{2/3}]^3 = 125. \tag{13}$$

Wherein the equality holds when $a = b = c = 2$.

6. **Let D, E, F be the points of contact of the incircle of an acute-angled triangle ABC with BC, CA, AB respectively. Let I_1, I_2, I_3 be the incentres of the triangles AFE, BDF, CED, respectively. Prove that the lines I_1D, I_2E, I_3F are concurrent.**

Solution

Observe that $\angle AFE = \angle AEF = 90^\circ - A/2$ and $\angle FDE = \angle AEF = 90^\circ - A/2$. Again $\angle EI_1F = 90^\circ + A/2$. Thus

$$\angle EI_1F + \angle FDE = 180^\circ.$$

Hence I_1 lies on the incircle. Also

$$\angle I_1FE = (1/2)\angle AFE = (1/2)\angle AEF = \angle I_1EF. \tag{14}$$

Thus $I_1E = I_1F$. But then they are equal chords of a circle and so they must subtend equal angles at the circumference. Therefore $\angle I_1DF = \angle I_1DE$ and so I_1D is the internal bisector of $\angle FDE$. Similarly we can show that I_2E and I_3F are internal bisectors of $\angle DEF$ and $\angle DFE$ respectively. Thus the three lines I_1D, I_2E, I_3F are concurrent at the incentre of triangle DEF.