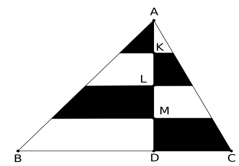
Solutions to RMO-2014 problems

1. Let ABC be a triangle and let AD be the perpendicular from A on to BC. Let K, L, M be points on AD such that $AK = KL = LM = MD$. If the sum of the areas of the shaded regions is equal to the sum of the areas of the unshaded regions, prove that $BD = DC$.



Solution: let $BD = 4x$, $DC = 4y$ and $AD = 4h$. Then the sum of the areas of the shaded regions is

$$\frac{1}{2}h\big(x + (y + 2y) + (2x + 3x) + (3y + 4y)\big) = \frac{h(6x + 10y)}{2}.$$

The sum of the areas of the unshaded regions is

$$\frac{1}{2}h\big(y + (x + 2x) + (2y + 3y) + (3x + 4x)\big) = \frac{h(10x + 6y)}{2}.$$

Therefore the given condition implies that

$$6x + 10y = 10x + 6y.$$

This gives $x = y$. Hence $BD = DC$.

2. Let $a_1, a_2, \ldots, a_{2n}$ be an arithmetic progression of positive real numbers with common difference d. Let
(i) $a_1^2 + a_3^2 + \cdots + a_{2n-1}^2 = x$, (ii) $a_2^2 + a_4^2 + \cdots + a_{2n}^2 = y$, and
(iii) $a_n + a_{n+1} = z$.
Express d in terms of x, y, z, n.

Solution: Observe that

$$y - x = (a_2^2 - a_1^2) + (a_4^2 - a_3^2) + \cdots + (a_{2n}^2 - a_{2n-1}^2).$$

The general difference is

$$a_{2k}^2 - a_{2k-1}^2 = \Big(a_{2k} + a_{2k-1}\Big)d = \Big(2a_1 + \big((2k-1) + (2k-2)\big)d\Big)d.$$

Therefore

$$y - x = \big(2na_1 + (1 + 2 + 3 + \cdots (2n-1))d\big)d = nd\big(2a_1 + (2n-1)d\big).$$

We also observe that

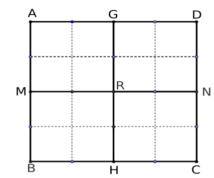
$$z = a_n + a_{n+1} = 2a_1 + (2n-1)d.$$

It follows that $y - x = ndz$. Hence $d = (y - x)/nz$.

3. Suppose for some positive integers r and s, the digits of 2^r is obtained by permuting the digits of 2^s in decimal expansion. Prove that $r = s$.

Solution: Suppose $s \leq r$. If $s < r$ then $2^s < 2^r$. Since the number of digits in 2^s and 2^r are the same, we have $2^r < 10 \times 2^s < 2^{s+4}$. Thus we have $2^s < 2^r < 2^{s+4}$ which gives $r = s+1$ or $s+2$ or $s+3$. Since 2^r is obtained from 2^s by permuting its digits, $2^r - 2^s$ is divisible by 9. If $r = s+1$, we see that $2^r - 2^s = 2^s$ and it is clearly not divisible by 9. Similarly, $2^{s+2} - 2^s = 3 \times 2^s$ and $2^{s+3} - 2^s = 7 \times 2^s$ and none of these is divisible by 9. We conclude that $s < r$ is not possible. Hence $r = s$.

4. Is it possible to write the numbers $17, 18, 19, \ldots, 32$ in a 4×4 grid of unit squares, with one number in each square, such that the product of the numbers in each 2×2 sub-grids $AMRG$, $GRND$, $MBHR$ and $RHCN$ is **divisible** by 16?



Solution: NO! If the product in each 2×2 sub-square is divisible by 16, then the product of all the numbers is divisible by $16 \times 16 \times 16 \times 16 = 2^{16}$. But it is easy to see that

$$17 \times 18 \times 19 \times \cdots \times 32 = 2^{15}k,$$

where k is an odd number. Hence the product of all the numbers in the grid is not divisible by 2^{16}.

5. Let ABC be an acute-angled triangle and let H be its ortho-centre. For any point P on the circum-circle of triangle ABC, let Q be the point of intersection of the line BH with the line AP. Show that there is a unique point X on the circum-circle of ABC such that for every point $P \neq A, B$, the circum-circle of HQP pass through X.

Solution: We consider two possibilities: Q lying between A and P; and P lying between A and Q. (See the figures.)

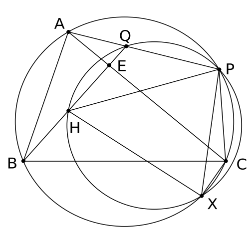
In the first case, we observe that

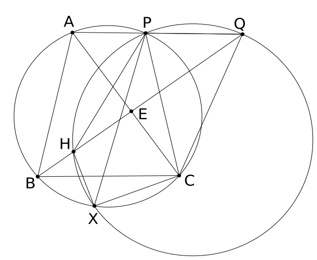
$$\angle HXC = \angle HXP + \angle PXC = \angle AQB + \angle PAC,$$

since Q, H, X, P are concyclic and P, A, X, C are also concyclic. Thus we get

$$\angle HXC = \angle AQE + \angle QAE = 90^\circ$$

because $BE \perp AC$.





In the second case, we have

$$\angle HXC = \angle HXP + \angle PXC = \angle HQP + \angle PAC;$$

the first follows from H, X, Q, P are concyclic; the second follows from the concyclicity of A, X, C, P. Again $BE \perp AC$ shows that $\angle HXC = 90°$.

Thus for any point $P \neq A, B$ on the circumcircle of ABC, the point X of intersection of the circumcircles of ABC and HPQ is such that $\angle HXC = 90°$. This means X is precisely the point of intersection of the circumcircles of HEC and ABC, which is independent of P.

6. Let $x_1, x_2, \ldots, x_{2014}$ be positive real numbers such that $\sum_{j=1}^{2014} x_j = 1$. Determine with proof the smallest constant K such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1.$$

Solution: Let us take the general case: $\{x_1, x_2, \ldots, x_n\}$ are positive real numbers such that $\sum_{k=1}^{n} x_k = 1$. Then

$$\sum_{k=1}^{n} \frac{x_k^2}{1 - x_k} = \sum_{k=1}^{n} \frac{x_k^2 - 1}{1 - x_k} + \sum_{k=1}^{n} \frac{1}{1 - x_k} = \sum_{k=1}^{n} (-1 - x_k) + \sum_{k=1}^{n} \frac{1}{1 - x_k}.$$

Now the first sum is $-n - 1$. We can estimate the second sum using AM-HM inequality:

$$\sum_{k=1}^{n} \frac{1}{1 - x_k} \geq \frac{n^2}{\sum_{k=1}^{n}(1 - x_k)} = \frac{n^2}{n - 1}.$$

Thus we obtain

$$\sum_{k=1}^{n} \frac{x_k^2}{1 - x_k} \geq -(1 + n) + \frac{n^2}{n - 1} = \frac{1}{n - 1}.$$

Here equality holds if and only if all x_j's are equal. Thus we get the smallest constant K such that

$$K \sum_{j=1}^{2014} \frac{x_j^2}{1 - x_j} \geq 1$$

to be $2014 - 1 = 2013$.

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