29^{th} Indian National Mathematical Olympiad-2014

February 02, 2014

1. In a triangle ABC, let D be a point on the segment BC such that $AB + BD = AC + CD$. Suppose that the points B, C and the centroids of triangles ABD and ACD lie on a circle. Prove that $AB = AC$.

   Solution. Let G_1, G_2 denote the centroids of triangles ABD and ACD. Then G_1, G_2 lie on the line parallel to BC that passes through the centriod of triangle ABC. Therefore BG_1G_2C is an isosceles trapezoid. Therefore it follows that $BG_1 = CG_2$. This proves that $AB^2+BD^2 = AC^2+CD^2$. Hence it follows that $AB\cdot BD = AC\cdot CD$. Therefore the sets $\{AB, BD\}$ and $\{AC, CD\}$ are the same (since they are both equal to the set of roots of the same polynomial). Note that if $AB = CD$ then $AC = BD$ and then $AB+AC = BC$, a contradiction. Therefore it follows that $AB = AC$.

2. Let n be a natural number. Prove that

$$\left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots \left[\frac{n}{n}\right] + \left[\sqrt{n}\right]$$

   is **even**. (Here $[x]$ denotes the largest integer smaller than or equal to x.)

   Solution. Let $f(n)$ denote the given equation. Then $f(1) = 2$ which is even. Now suppose that $f(n)$ is even for some $n \geq 1$. Then

$$\begin{aligned} f(n+1) &= \left[\frac{n+1}{1}\right] + \left[\frac{n+1}{2}\right] + \left[\frac{n+1}{3}\right] + \cdots \left[\frac{n+1}{n+1}\right] + \left[\sqrt{n+1}\right] \\ &= \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \left[\frac{n}{3}\right] + \cdots \left[\frac{n}{n}\right] + \left[\sqrt{n+1}\right] + \sigma(n+1)\,, \end{aligned}$$

   where $\sigma(n+1)$ denotes the number of positive divisors of $n+1$. This follows from $\left[\frac{n+1}{k}\right] = \left[\frac{n}{k}\right] + 1$ if k divides $n+1$, and $\left[\frac{n+1}{k}\right] = \left[\frac{n}{k}\right]$ otherwise. Note that $\left[\sqrt{n+1}\right] = \left[\sqrt{n}\right]$ unless $n+1$ is a square, in which case $\left[\sqrt{n+1}\right] = \left[\sqrt{n}\right] + 1$. On the other hand $\sigma(n+1)$ is odd if and only if $n+1$ is a square. Therefore it follows that $f(n+1) = f(n) + 2l$ for some integer l. This proves that $f(n+1)$ is even.

   Thus it follows by induction that $f(n)$ is even for all natural number n.

3. Let a, b be natural numbers with $ab > 2$. Suppose that the sum of their greatest common divisor and least common multiple is divisible by $a+b$. Prove that the quotient is at most $(a+b)/4$. When is this quotient exactly equal to $(a+b)/4$?

   Solution. Let g and l denote the greatest common divisor and the least common multiple, respectively, of a and b. Then $gl = ab$. Therefore $g + l \leq ab + 1$. Suppose that $(g + l)/(a + b) > (a + b)/4$. Then we have $ab+1 > (a+b)^2/4$, so we get $(a-b)^2 < 4$. Assuming, $a \geq b$ we either have $a = b$ or $a = b+1$. In the former case, $g = l = a$ and the quotient is $(g + l)/(a + b) = 1 \leq (a + b)/4$. In the latter case, $g = 1$ and $l = b(b + 1)$ so we get that $2b+1$ divides $b^2 + b+1$. Therefore $2b+1$ divides $4(b^2 + b + 1) - (2b + 1)^2 = 3$ which implies that $b = 1$ and $a = 2$, a contradiction to the given assumption that $ab > 2$. This shows that $(g + l)/(a + b) \leq (a + b)/4$. Note that for the equality to hold, we need that either $a = b = 2$ or, $(a - b)^2 = 4$ and $g = 1, l = ab$. The latter case happens if and only if a and b are two consecutive odd numbers. (If $a = 2k + 1$ and $b = 2k - 1$ then $a + b = 4k$ divides $ab + 1 = 4k^2$ and the quotient is precisely $(a + b)/4$.)

4. Written on a blackboard is the polynomial $x^2 + x + 2014$. Calvin and Hobbes take turns alternatively (starting with Calvin) in the following game. During his turn, Calvin should either increase or decrease the coefficient of x by 1. And during his turn, Hobbes should either increase or decrease the constant coefficient by 1. Calvin wins if at any point of time the polynomial on the blackboard at that instant has integer roots. Prove that Calvin has a winning strategy.

   Solution. For $i \geq 0$, let $f_i(x)$ denote the polynomial on the blackboard after Hobbes' i-th turn. We let Calvin decrease the coefficient of x by 1. Therefore $f_{i+1}(2) = f_i(2) - 1$ or $f_{i+1}(2) = f_i(2) - 3$ (depending on whether Hobbes increases or decreases the constant term). So for some i, we have $0 \leq f_i(2) \leq 2$. If $f_i(2) = 0$ then Calvin has won the game. If $f_i(2) = 2$ then Calvin wins the game by reducing the coefficient of x by 1. If $f_i(2) = 1$ then $f_{i+1}(2) = 0$ or $f_{i+1}(2) = -2$. In the former case, Calvin has won the game and in the latter case Calvin wins the game by increasing the coefficient of x by 1.

5. In an acute-angled triangle ABC, a point D lies on the segment BC. Let O_1, O_2 denote the circumcentres of triangles ABD and ACD, respectively. Prove that the line joining the circumcentre of triangle ABC and the orthocentre of triangle O_1O_2D is parallel to BC.

   Solution. Without loss of generality assume that $\angle ADC \geq 90°$. Let O denote the circumcenter of triangle ABC and K the orthocentre of triangle O_1O_2D. We shall first show that the points O and K lie on the circumcircle of triangle AO_1O_2. Note that circumcircles of triangles ABD and ACD pass through the points A and D, so AD is perpendicular to O_1O_2 and, triangle AO_1O_2 is congruent to triangle DO_1O_2. In particular, $\angle AO_1O_2 = \angle O_2O_1D = \angle B$ since O_2O_1 is the perpendicular bisector of AD. On the other hand since OO_2 is the perpendicular bisector of AC it follows that $\angle AOO_2 = \angle B$. This shows that O lies on the circumcircle of triangle AO_1O_2. Note also that, since AD is perpendicular to O_1O_2, we have $\angle O_2KA = 90° - \angle O_1O_2K = \angle O_2O_1D = \angle B$. This proves that K also lies on the circumcircle of triangle AO_1O_2.

   Therefore $\angle AKO = 180° - \angle AO_2O = \angle ADC$ and hence OK is parallel to BC.

   Remark. The result is true even for an obtuse-angled triangle.

6. Let n be a natural number and $X = \{1, 2, \ldots, n\}$. For subsets A and B of X we define $A\Delta B$ to be the set of all those elements of X which belong to exactly one of A and B. Let $\mathcal{F}$ be a collection of subsets of X such that for any two distinct elements A and B in $\mathcal{F}$ the set $A\Delta B$ has at least two elements. Show that $\mathcal{F}$ has at most 2^{n-1} elements. Find all such collections $\mathcal{F}$ with 2^{n-1} elements.

   Solution. For each subset A of $\{1, 2, \ldots, n-1\}$, we pair it with $A \cup \{n\}$. Note that for any such pair (A, B) not both A and B can be in $\mathcal{F}$. Since there are 2^{n-1} such pairs it follows that $\mathcal{F}$ can have at most 2^{n-1} elements.

   We shall show by induction on n that if $|\mathcal{F}| = 2^{n-1}$ then $\mathcal{F}$ contains either all the subsets with odd number of elements or all the subsets with even number of elements. The result is easy to see for $n = 1$. Suppose that the result is true for $n = m - 1$. We now consider the case $n = m$. Let $\mathcal{F}_1$ be the set of those elements in $\mathcal{F}$ which contain m and $\mathcal{F}_2$ be the set of those elements which do not contain m. By induction, $\mathcal{F}_2$ can have at most 2^{m-2} elements. Further, for each element A of $\mathcal{F}_1$ we consider $A \setminus \{m\}$. This new collection also satisfies the required property, so it follows that $\mathcal{F}_1$ has at most 2^{m-2} elements. Thus, if $|\mathcal{F}| = 2^{m-1}$ then it follows that $|\mathcal{F}_1| = |\mathcal{F}_2| = 2^{m-2}$. Further, by induction hypothesis, $\mathcal{F}_2$ contains all those subsets of $\{1, 2, \ldots, m-1\}$ with (say) even number of elements. It then follows that $\mathcal{F}_1$ contains all those subsets of $\{1, 2, \ldots, m\}$ which contain the element m and which contains an even number of elements. This proves that $\mathcal{F}$ contains either all the subsets with odd number of elements or all the subsets by even number of elements.

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