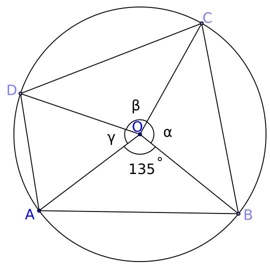
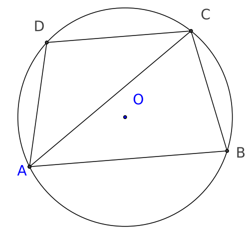
Problems and Solutions: INMO-2012

1. Let $ABCD$ be a quadrilateral inscribed in a circle. Suppose $AB = \sqrt{2+\sqrt{2}}$ and AB subtends $135°$ at the centre of the circle. Find the maximum possible area of $ABCD$.





Solution: Let O be the centre of the circle in which $ABCD$ is inscribed and let R be its radius. Using cosine rule in triangle AOB, we have

$$2+\sqrt{2} = 2R^2(1-\cos 135°) = R^2(2+\sqrt{2}).$$

Hence $R = 1$.

Consider quadrilateral $ABCD$ as in the second figure above. Join AC. For $[ADC]$ to be maximum, it is clear that D should be the mid-point of the arc AC so that its distance from the segment AC is maximum. Hence $AD = DC$ for $[ABCD]$ to be maximum. Similarly, we conclude that $BC = CD$. Thus $BC = CD = DA$ which fixes the quadrilateral $ABCD$. Therefore each of the sides BC, CD, DA subtends equal angles at the centre O.

Let $\angle BOC = \alpha$, $\angle COD = \beta$ and $\angle DOA = \gamma$. Observe that

$$[ABCD] = [AOB]+[BOC]+[COD]+[DOA] = \frac{1}{2}\sin 135° + \frac{1}{2}(\sin\alpha + \sin\beta + \sin\gamma).$$

Now $[ABCD]$ has maximum area if and only if $\alpha = \beta = \gamma = (360° - 135°)/3 = 75°$. Thus

$$[ABCD] = \frac{1}{2}\sin 135° + \frac{3}{2}\sin 75° = \frac{1}{2}\left(\frac{1}{\sqrt{2}} + 3\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = \frac{5+3\sqrt{3}}{4\sqrt{2}}.$$

Alternatively, we can use Jensen's inequality. Observe that α, β, γ are all less than $180°$. Since $\sin x$ is concave on $(0,\pi)$, Jensen's inequality gives

$$\frac{\sin\alpha + \sin\beta + \sin\gamma}{3} \leq \sin\left(\frac{\alpha+\beta+\gamma}{3}\right) = \sin 75°.$$

Hence

$$[ABCD] \leq \frac{1}{2\sqrt{2}} + \frac{3}{2}\sin 75° = \frac{5+3\sqrt{3}}{4\sqrt{2}},$$

with equality if and only if $\alpha = \beta = \gamma = 75°$.

2. Let $p_1 < p_2 < p_3 < p_4$ and $q_1 < q_2 < q_3 < q_4$ be two sets of prime numbers such that $p_4 - p_1 = 8$ and $q_4 - q_1 = 8$. Suppose $p_1 > 5$ and $q_1 > 5$. Prove that 30 divides $p_1 - q_1$.

Solution: Since $p_4 - p_1 = 8$, and no prime is even, we observe that $\{p_1, p_2, p_3, p_4\}$ is a subset of $\{p_1, p_1+2, p_1+4, p_1+6, p_1+8\}$. Moreover p_1 is larger than 3. If $p_1 \equiv 1 \pmod 3$, then $p_1 + 2$ and $p_1 + 8$ are divisible by 3. Hence we do not get 4 primes in the set $\{p_1, p_1+2, p_1+4, p_1+6, p_1+8\}$. Thus $p_1 \equiv 2 \pmod 3$ and $p_1 + 4$ is not a prime. We get $p_2 = p_1 + 2, p_3 = p_1 + 6, p_4 = p_1 + 8$.

Consider the remainders of p_1, p_1+2, p_1+6, p_1+8 when divided by 5. If $p_1 \equiv 2 \pmod 5$, then $p_1 + 8$ is divisible by 5 and hence is not a prime. If $p_1 \equiv 3 \pmod 5$, then $p_1 + 2$ is divisible by 5. If $p_1 \equiv 4 \pmod 5$, then $p_1 + 6$ is divisible by 5. Hence the only possibility is $p_1 \equiv 1 \pmod 5$.

Thus we see that $p_1 \equiv 1 \pmod 2$, $p_1 \equiv 2 \pmod 3$ and $p_1 \equiv 1 \pmod 5$. We conclude that $p_1 \equiv 11 \pmod{30}$.

Similarly $q_1 \equiv 11 \pmod{30}$. It follows that 30 divides $p_1 - q_1$.

3. Define a sequence $\langle f_0(x), f_1(x), f_2(x), \ldots \rangle$ of functions by

$$f_0(x) = 1, \quad f_1(x) = x, \quad \big(f_n(x)\big)^2 - 1 = f_{n+1}(x) f_{n-1}(x), \text{ for } n \geq 1.$$

Prove that each $f_n(x)$ is a polynomial with integer coefficients.

Solution: Observe that

$$f_n^2(x) - f_{n-1}(x) f_{n+1}(x) = 1 = f_{n-1}^2(x) - f_{n-2}(x) f_n(x).$$

This gives

$$f_n(x)\Big(f_n(x) + f_{n-2}(x)\Big) = f_{n-1}\Big(f_{n-1}(x) + f_{n+1}(x)\Big).$$

We write this as

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{f_{n-2}(x) + f_n(x)}{f_{n-1}(x)}.$$

Using induction, we get

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{f_0(x) + f_2(x)}{f_1(x)}.$$

Observe that

$$f_2(x) = \frac{f_1^2(x) - 1}{f_0(x)} = x^2 - 1.$$

Hence

$$\frac{f_{n-1}(x) + f_{n+1}(x)}{f_n(x)} = \frac{1 + (x^2 - 1)}{x} = x.$$

Thus we obtain

$$f_{n+1}(x) = x f_n(x) - f_{n-1}(x).$$

Since $f_0(x)$, $f_1(x)$ and $f_2(x)$ are polynomials with integer coefficients, induction again shows that $f_n(x)$ is a polynomial with integer coefficients.

Note: We can get $f_n(x)$ explicitly:

$$f_n(x) = x^n - \binom{n-1}{1}x^{n-2} + \binom{n-2}{2}x^{n-4} - \binom{n-3}{3}x^{n-6} + \cdots$$

4. Let ABC be a triangle. An interior point P of ABC is said to be **good** if we can find exactly 27 rays emanating from P intersecting the sides of the triangle ABC such that the triangle is divided by these rays into 27 *smaller triangles of equal area*. Determine the number of **good** points for a given triangle ABC.

Solution: Let P be a good point. Let l, m, n be respetively the number of parts the sides BC, CA, AB are divided by the rays starting from P. Note that a ray must pass through each of the vertices the triangle ABC; otherwise we get some quadrilaterals.

Let h_1 be the distance of P from BC. Then h_1 is the height for all the triangles with their bases on BC. Equality of areas implies that all these bases have equal length. If we denote this by x, we get $lx = a$. Similarly, taking y and z as the lengths of the bases of triangles on CA and AB respectively, we get $my = b$ and $nz = c$. Let h_2 and h_3 be the distances of P from CA and AB respectively. Then

$$h_1 x = h_2 y = h_3 z = \frac{2\Delta}{27},$$

where Δ denotes the area of the triangle ABC. These lead to

$$h_1 = \frac{2\Delta}{27}\frac{l}{a}, \quad h_1 = \frac{2\Delta}{27}\frac{m}{b}, \quad h_1 = \frac{2\Delta}{27}\frac{n}{c}.$$

But

$$\frac{2\Delta}{a} = h_a, \quad \frac{2\Delta}{b} = h_b, \quad \frac{2\Delta}{c} = h_c.$$

Thus we get

$$\frac{h_1}{h_a} = \frac{l}{27}, \quad \frac{h_2}{h_b} = \frac{m}{27}, \quad \frac{h_3}{h_c} = \frac{n}{27}.$$

However, we also have

$$\frac{h_1}{h_a} = \frac{[PBC]}{\Delta}, \quad \frac{h_2}{h_b} = \frac{[PCA]}{\Delta}, \quad \frac{h_3}{h_c} = \frac{[PAB]}{\Delta}.$$

Adding these three relations,

$$\frac{h_1}{h_a} + \frac{h_2}{h_b} + \frac{h_3}{h_c} = 1.$$

Thus

$$\frac{l}{27} + \frac{m}{27} + \frac{n}{27} = \frac{h_1}{h_a} + \frac{h_2}{h_b} + \frac{h_3}{h_c} = 1.$$

We conclude that $l+m+n=27$. Thus every **good** point P determines a partition (l,m,n) of 27 such that there are l, m, n equal segments respectively on BC, CA, AB.

Conversely, take any partition (l,m,n) of 27. Divide BC, CA, AB respectively in to l, m, n equal parts. Define

$$h_1 = \frac{2l\Delta}{27a}, \quad h_2 = \frac{2m\Delta}{27b}.$$

Draw a line parallel to BC at a distance h_1 from BC; draw another line parallel to CA at a distance h_2 from CA. Both lines are drawn such that they intersect at a point P inside the triangle ABC. Then

$$[PBC] = \frac{1}{2}ah_1 = \frac{l\Delta}{27}, \quad [PCA] = \frac{m\Delta}{27}.$$

Hence

$$[PAB] = \frac{n\Delta}{27}.$$

This shows that the distance of P from AB is

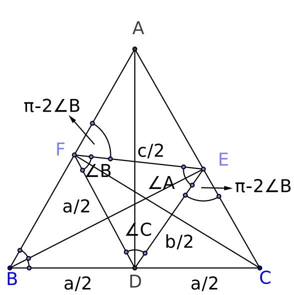
$$h_3 = \frac{2n\Delta}{27c}.$$

Therefore each traingle with base on CA has area $\dfrac{\Delta}{27}$. We conclude that all the triangles which partitions ABC have equal areas. Hence P is a **good** point.

Thus the number of **good** points is equal to the number of positive integral solutions of the equation $l+m+n=27$. This is equal to

$$\binom{26}{2} = 325.$$

5. Let ABC be an acute-angled triangle, and let D, E, F be points on BC, CA, AB respectively such that AD is the median, BE is the internal angle bisector and CF is the altitude. Suppose $\angle FDE = \angle C$, $\angle DEF = \angle A$ and $\angle EFD = \angle B$. Prove that ABC is equilateral.



Solution: Since ΔBFC is right-angled at F, we have $FD = BD = CD = a/2$. Hence $\angle BFD = \angle B$. Since $\angle EFD = \angle B$, we have $\angle AFE = \pi - 2\angle B$. Since $\angle DEF = \angle A$, we also get $\angle CED = \pi - 2\angle B$. Applying sine rule in ΔDEF, we have

$$\frac{DF}{\sin A} = \frac{FE}{\sin C} = \frac{DE}{\sin B}.$$

Thus we get $FE = c/2$ and $DE = b/2$. Sine rule in ΔCED gives

$$\frac{DE}{\sin C} = \frac{CD}{\sin(\pi - 2B)}.$$

Thus $(b/\sin C) = (a/2\sin B\cos B)$. Solving for $\cos B$, we have

$$\cos B = \frac{a\sin c}{2b\sin B} = \frac{ac}{2b^2}.$$

Similarly, sine rule in ΔAEF gives

$$\frac{EF}{\sin A} = \frac{AE}{\sin(\pi - 2B)}.$$

This gives (since $AE = bc/(a+c)$), as earlier,

$$\cos B = \frac{a}{a+c}.$$

Comparing the two values of $\cos B$, we get $2b^2 = c(a+c)$. We also have

$$c^2 + a^2 - b^2 = 2ca\cos B = \frac{2a^2c}{a+c}.$$

Thus

$$4a^2c = (a+c)(2c^2 + 2a^2 - 2b^2) = (a+c)(2c^2 + 2a^2 - c(a+c)).$$

This reduces to $2a^3 - 3a^2c + c^3 = 0$. Thus $(a-c)^2(2a+c) = 0$. We conclude that $a = c$. Finally

$$2b^2 = c(a+c) = 2c^2.$$

We thus get $b = c$ and hence $a = c = b$. This shows that ΔABC is equilateral.

6. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a function satisfying $f(0) \neq 0$, $f(1) = 0$ and

   (i) $f(xy) + f(x)f(y) = f(x) + f(y)$;

   (ii) $\big(f(x-y) - f(0)\big)f(x)f(y) = 0$,

   for all $x, y \in \mathbb{Z}$, simultaneously.

   (a) Find the set of all possible values of the function f.

   (b) If $f(10) \neq 0$ and $f(2) = 0$, find the set of all integers n such that $f(n) \neq 0$.

Solution: Setting $y = 0$ in the condition (ii), we get

$$\big(f(x) - f(0)\big)f(x) = 0,$$

for all x (since $f(0) \neq 0$). Thus either $f(x) = 0$ or $f(x) = f(0)$, for all $x \in \mathbb{Z}$. Now taking $x = y = 0$ in (i), we see that $f(0) + f(0)^2 = 2f(0)$. This shows that $f(0) = 0$ or $f(0) = 1$. Since $f(0) \neq 0$, we must have $f(0) = 1$. We conclude that

$$\text{either } f(x) = 0 \text{ or } f(x) = 1 \text{ for each } x \in \mathbb{Z}.$$

This shows that the set of all possible value of $f(x)$ is $\{0, 1\}$. This completes (a).

Let $S = \{n \in \mathbb{Z} \big| f(n) \neq 0\}$. Hence we must have $S = \{n \in \mathbb{Z} \big| f(n) = 1\}$ by (a). Since $f(1) = 0$, 1 is not in S. And $f(0) = 1$ implies that $0 \in S$. Take any $x \in \mathbb{Z}$ and $y \in S$. Using (ii), we get

$$f(xy) + f(x) = f(x) + 1.$$

This shows that $xy \in S$. If $x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ are such that $xy \in S$, then (ii) gives

$$1 + f(x)f(y) = f(x) + f(y).$$

Thus $\big(f(x) - 1\big)\big(f(y) - 1\big) = 0$. It follows that $f(x) = 1$ or $f(y) = 1$; i.e., either $x \in S$ or $y \in S$. We also observe from (ii) that $x \in S$ and $y \in S$ implies that $f(x - y) = 1$ so that $x - y \in S$. Thus S has the properties:

(A) $x \in \mathbb{Z}$ and $y \in S$ implies $xy \in S$;

(B) $x, y \in \mathbb{Z}$ and $xy \in S$ implies $x \in S$ or $y \in S$;

(C) $x, y \in S$ implies $x - y \in S$.

Now we know that $f(10) \neq 0$ and $f(2) = 0$. Hence $f(10) = 1$ and $10 \in S$; and $2 \notin S$. Writing $10 = 2 \times 5$ and using (B), we conclude that $5 \in S$ and $f(5) = 1$. Hence $f(5k) = 1$ for all $k \in \mathbb{Z}$ by (A).

Suppose $f(5k + l) = 1$ for some l, $1 \leq l \leq 4$. Then $5k + l \in S$. Choose $u \in \mathbb{Z}$ such that $lu \equiv 1 \pmod 5$. We have $(5k + l)u \in S$ by (A). Moreover, $lu = 1 + 5m$ for some $m \in \mathbb{Z}$ and

$$(5k + l)u = 5ku + lu = 5ku + 5m + 1 = 5(ku + m) + 1.$$

This shows that $5(ku+m)+1 \in S$. However, we know that $5(ku+m) \in S$. By (C), $1 \in S$ which is a contradiction. We conclude that $5k + l \notin S$ for any l, $1 \leq l \leq 4$. Thus

$$S = \{5k \big| k \in \mathbb{Z}\}.$$

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