Problems and Solutions of INMO-2007

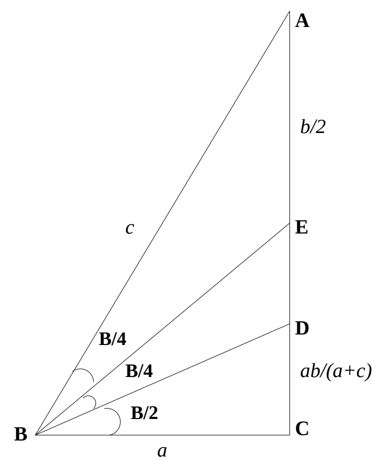
1. In a triangle ABC right-angled at C, the median through B bisects the angle between BA and the bisector of $\angle B$. Prove that

$$\frac{5}{2} < \frac{AB}{BC} < 3.$$

Solution 1:

Since E is the mid-point of AC, we have $AE = EC = b/2$. Since BD bisects $\angle ABC$, we also know that $CD = ab/(a+c)$. Since BE bisects $\angle ABD$, we also have

$$\frac{BD^2}{BA^2} = \frac{DE^2}{EA^2}.$$



However,

$$BD^2 = BC^2 + CD^2 = a^2 + \frac{a^2b^2}{(a+c)^2},$$
$$DE^2 = \left(\frac{b}{2} - \frac{ab}{a+c}\right)^2.$$

Using these in the above expression and simplifying, we get

$$a^2\{(a+c)^2 + b^2\} = c^2(c-a)^2.$$

Using $c^2 = a^2 + b^2$ and eliminating b, we obtain

$$c^3 - 2ac^2 - a^2c - 2a^3 = 0.$$

Introducing $t = c/a$, this reduces to a cubic equation;

$$t^3 - 2t^2 - t - 2 = 0.$$

Consider the function $f(t) = t^3 - 2t^2 - t - 2$ for $t > 0$ (as c/a is positive). For $0 < t \le 2$, we see that $f(t) = t^2(t-2) - t - 2 < 0$. We also observe that $f(t) = (t-2)(t^2-1) - 4$ is strictly increasing on $(2, \infty)$. It is easy to compute

$$f(5/2) = -\frac{11}{8} < 0, \quad \text{and} \quad f(3) = 4 > 0.$$

Hence there is a unique value of t in the interval $(5/2, 3)$ such that $f(t) = 0$. We conclude that

$$\frac{5}{2} < \frac{c}{a} < 3.$$

Solution 2: Let us take $\angle B/4 = \theta$. Then $\angle EBC = \angle DBE = \theta$ and $\angle CBD = 2\theta$.Using sine rule in triangles BEA and BEC, we get

$$\frac{BE}{\sin A} = \frac{AE}{\sin\theta},$$
$$\frac{BE}{\sin 90^\circ} = \frac{CE}{\sin 3\theta}.$$

Since $AE = CE$, we obtain $\sin 3\theta \sin A = \sin\theta$. However $A = 90^\circ - 4\theta$. Thus we get $\sin 3\theta \cos 4\theta = \sin\theta$. Note that

$$\frac{c}{a} = \frac{1}{\cos 4\theta} = \frac{\sin 3\theta}{\sin\theta} = 3 - 4\sin^2\theta.$$

This shows that $c/a < 3$. Using $c/a = 3 - 4\sin^2\theta$, it is easy to compute $\cos 2\theta = \big((c/a) - 1\big)/2$. Hence

$$\frac{a}{c} = \cos 4\theta = \frac{1}{2}\left(\frac{c}{a} - 1\right)^2 - 1.$$

Suppose $c/a \le 5/2$. Then $\big((c/a) - 1\big)^2 \le 9/4$ and $a/c \ge 2/5$. Thus

$$\frac{2}{5} \le \frac{a}{c} = \frac{1}{2}\left(\frac{c}{a} - 1\right)^2 - 1 \le \frac{9}{8} - 1 = \frac{1}{8},$$

which is absurd. We conclude that $c/a > 5/2$.

2. Let n be a natural number such that $n = a^2 + b^2 + c^2$, for some natural numbers a, b, c. Prove that

$$9n = \big(p_1a + q_1b + r_1c\big)^2 + \big(p_2a + q_2b + r_2c\big)^2 + \big(p_3a + q_3b + r_3c\big)^2,$$

where p_j's, q_j's, r_j's are all **nonzero** integers. Further, if 3 does **not** divide at least one of a, b, c, prove that $9n$ can be expressed in the form $x^2 + y^2 + z^2$, where x, y, z are natural numbers **none** of which is divisible by 3.

Solution: It can be easily seen that

$$9n = (2b + 2c - a)^2 + (2c + 2a - b)^2 + (2a + 2b - c)^2.$$

Thus we can take $p_1 = p_2 = p_3 = 2$, $q_1 = q_2 = q_3 = 2$ and $r_1 = r_2 = r_3 = -1$. Suppose 3 does not divide $\gcd(a, b, c)$. Then 3 does divide at least one of a, b, c; say 3 does not divide a. Note that each of $2b + 2c - a$, $2c + 2a - b$ and $2a + 2b - c$ is either divisible by 3 or none of them is divisible by 3, as the difference of any two sums is always divisible by 3. If 3 does not divide $2b + 2c - a$, then we have the required representation. If 3 divides $2b + 2c - a$, then 3 does not divide $2b + 2c + a$. On the other hand, we also note that

$$9n = (2b + 2c + a)^2 + (2c - 2a - b)^2 + (-2a + 2b - c)^2 = x^2 + y^2 + z^2,$$

where $x = 2b + 2c + a$, $y = 2c - 2a - b$ and $z = -2a + 2b - c$. Since $x - y = 3(b + a)$ and 3 does not divide x, it follows that 3 does not divide y as well. Similarly, we conclude that 3 does not divide z.

3. Let m and n be positive integers such that the equation $x^2 - mx + n = 0$ has real roots α and β. Prove that α and β are integers if and only if $[m\alpha] + [m\beta]$ is the square of an integer. (Here $[x]$ denotes the largest integer not exceeding x.)

Solution: If α and β are both integers, then

$$[m\alpha] + [m\beta] = m\alpha + m\beta = m\big(\alpha + \beta\big) = m^2.$$

This proves one implication.

Observe that $\alpha+\beta=m$ and $\alpha\beta=n$. We use the property of integer function: $x-1<[x]\leq x$ for any real number x. Thus

$$m^2-2=m\big(\alpha+\beta\big)-2=m\alpha-1+m\beta-1<[m\alpha]+[m\beta]\leq m\big(\alpha+\beta\big)=m^2.$$

Since m and n are positive integers, both α and β must be positive. If $m\geq 2$, we observe that there is no square between m^2-2 and m^2. Hence, either $m=1$ or $[m\alpha]+[m\beta]=m^2$. If $m=1$, then $\alpha+\beta=1$ implies that both α and β are positive reals smaller than 1. Hence $n=\alpha\beta$ cannot be a positive integer. We conclude that $[m\alpha]+[m\beta]=m^2$. Putting $m=\alpha+\beta$ in this relation, we get

$$\big[\alpha^2+n\big]+\big[\beta^2+n\big]=\big(\alpha+\beta\big)^2.$$

Using $[x+k]=[x]+k$ for any real number x and integer k, this reduces to

$$\big[\alpha^2\big]+\big[\beta^2\big]=\alpha^2+\beta^2.$$

This shows that α^2 and β^2 are both integers. On the other hand,

$$\alpha^2-\beta^2=\big(\alpha+\beta\big)\big(\alpha-\beta\big)=m\big(\alpha-\beta\big).$$

Thus

$$\big(\alpha-\beta\big)=\frac{\alpha^2-\beta^2}{m},$$

is a rational number. Since $\alpha+\beta=m$ is a rational number, it follows that both α and β are rational numbers. However, both α^2 and β^2 are integers. Hence each of α and β is an integer.

4. Let $\sigma=(a_1,a_2,a_3,\ldots,a_n)$ be a permutation of $(1,2,3,\ldots,n)$. A pair (a_i,a_j) is said to correspond to an inversion of σ, if $i<j$ but $a_i>a_j$. (Example: In the permutation $(2,4,5,3,1)$, there are 6 inversions corresponding to the pairs $(2,1)$, $(4,3)$, $(4,1)$, $(5,3)$, $(5,1)$, $(3,1)$.) How many permutations of $(1,2,3,\ldots n)$, $(n\geq 3)$, have exactly **two** inversions?

Solution: In a permutation of $(1,2,3,\ldots,n)$, two inversions can occur in only one of the following two ways:

(A) Two disjoint consecutive pairs are interchanged:

$$\begin{aligned}&(1,2,3,j-1,j,j+1,j+2\ldots k-1,k,k+1,k+2,\ldots,n)\\&\quad\longrightarrow(1,2,\ldots j-1,j+1,j,j+2,\ldots,k-1,k+1,k,k+2,\ldots,n).\end{aligned}$$

(B) Each block of three consecutive integers can be permuted in any of the following 2 ways;

$$\begin{aligned}(1,2,3,\ldots k,k+1,k+2,\ldots,n)&\longrightarrow(1,2,\ldots,k+2,k,k+1,\ldots,n);\\(1,2,3,\ldots k,k+1,k+2,\ldots,n)&\longrightarrow(1,2,\ldots,k+1,k+2,k,\ldots,n).\end{aligned}$$

Consider case (A). For $j=1$, there are $n-3$ possible values of k; for $j=2$, there are $n-4$ possibilities for k and so on. Thus the number of permutations with two inversions of this type is

$$1+2+\cdots+(n-3)=\frac{(n-3)(n-2)}{2}.$$

In case (B), we see that there are $n-2$ permutations of each type, since k can take values from 1 to $n-2$. Hence we get $2(n-2)$ permutations of this type.

Finally, the number of permutations with **two** inversions is

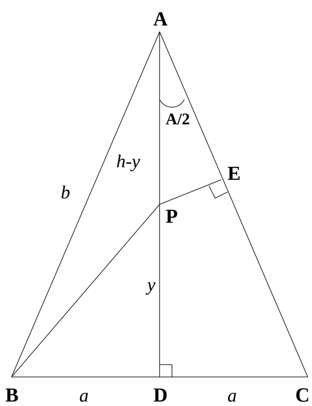
$$\frac{(n-3)(n-2)}{2} + 2(n-2) = \frac{(n+1)(n-2)}{2}.$$

5. Let ABC be a triangle in which $AB = AC$. Let D be the mid-point of BC and P be a point on AD. Suppose E is the foot of perpendicular from P on AC. If $\frac{AP}{PD} = \frac{BP}{PE} = \lambda$, $\frac{BD}{AD} = m$ and $z = m^2(1+\lambda)$, prove that

$$z^2 - \left(\lambda^3 - \lambda^2 - 2\right)z + 1 = 0.$$

Hence show that $\lambda \geq 2$ and $\lambda = 2$ if and only if ABC is equilateral.

Solution:



Let $AD = h$, $PD = y$ and $BD = DC = a$. We observe that $BP^2 = a^2 + y^2$. Moreover,

$$PE = PA \sin \angle DAC = (h-y)\frac{DC}{AC} = \frac{a(h-y)}{b},$$

where $b = AC = AB$. Using $AP/PD = (h-y)/y$, we obtain $y = h/(1+\lambda)$. Thus

$$\lambda^2 = \frac{BP^2}{PE^2} = \frac{(a^2+y^2)b^2}{(h-y)^2a^2}.$$

But $(h-y) = \lambda y = \lambda h/(1+\lambda)$ and $b^2 = a^2 + h^2$. Thus we obtain

$$\lambda^4 = \frac{\left(a^2(1+\lambda)^2 + h^2\right)(a^2+h^2)}{a^2h^2}.$$

Using $m = a/h$ and $z = m^2(1+\lambda)$, this simplifies to

$$z^2 - z(\lambda^3 - \lambda^2 - 2) + 1 = 0.$$

Dividing by z, this gives

$$z + \frac{1}{z} = \lambda^3 - \lambda^2 - 2.$$

However $z + (1/z) \geq 2$ for any positive real number z. Thus $\lambda^3 - \lambda^2 - 4 \geq 0$. This may be written in the form $\left(\lambda - 2\right)\left(\lambda^2 + \lambda + 2\right) \geq 0$. But $\lambda^2 + \lambda + 2 > 0$. (For example, one may check that its discriminant is negative.) Hence $\lambda \geq 2$. If $\lambda = 2$, then $z + (1/z) = 2$ and hence $z = 1$. This gives $m^2 = 1/3$ or $\tan(A/2) = m = 1/\sqrt{3}$. Thus $A = 60°$ and hence ABC is equilateral.

Conversely, if triangle ABC is equilateral, then $m = \tan(A/2) = 1/\sqrt{3}$ and hence $z = (1+\lambda)/3$. Substituting this in the equation satisfied by z, we obtain

$$\left(1+\lambda\right)^2 - 3\left(1+\lambda\right)\left(\lambda^3 - \lambda^2 - 2\right) + 9 = 0.$$

This may be written in the form $\left(\lambda - 2\right)\left(3\lambda^3 + 6\lambda^2 + 8\lambda + 8\right) = 0$. Here the second factor is positive because $\lambda > 0$. We conclude that $\lambda = 2$.

6. If x, y, z are positive real numbers, prove that

$$(x+y+z)^2\big(yz+zx+xy\big)^2 \le 3\big(y^2+yz+z^2\big)\big(z^2+zx+x^2\big)\big(x^2+xy+y^2\big).$$

Solution 1: We begin with the observation that

$$x^2+xy+y^2 = \frac{3}{4}(x+y)^2 + \frac{1}{4}(x-y)^2 \ge \frac{3}{4}(x+y)^2,$$

and similar bounds for y^2+yz+z^2, z^2+zx+x^2. Thus

$$3\big(x^2+xy+y^2\big)\big(y^2+yz+z^2\big)\big(z^2+zx+x^2\big) \ge \frac{81}{64}\big(x+y\big)^2\big(y+z\big)^2\big(z+x\big)^2.$$

Thus it is sufficient to prove that

$$\big(x+y+z\big)\big(xy+yz+zx\big) \le \frac{9}{8}\big(x+y\big)\big(y+z\big)\big(z+x\big).$$

Equivalently, we need to prove that

$$8\big(x+y+z\big)\big(xy+yz+zx\big) \le 9\big(x+y\big)\big(y+z\big)\big(z+x\big).$$

However, we note that

$$\big(x+y\big)\big(y+z\big)\big(z+x\big) = \big(x+y+z\big)\big(yz+zx+xy\big) - xyz.$$

Thus the required inequality takes the form

$$\big(x+y\big)\big(y+z\big)\big(z+x\big) \ge 8xyz.$$

This follows from AM-GM inequalities;

$$x+y \ge 2\sqrt{xy}, \quad y+z \ge 2\sqrt{yz}, \quad z+x \ge 2\sqrt{zx}.$$

Solution 2: Let us introduce $x+y=c$, $y+z=a$ and $z+x=b$. Then a, b, c are the sides of a triangle. If $s=(a+b+c)/2$, then it is easy to calculate $x=s-a$, $y=s-b$, $z=s-c$ and $x+y+z=s$. We also observe that

$$x^2+xy+y^2 = (x+y)^2 - xy = c^2 - \frac{1}{4}(c+a-b)(c+b-a) = \frac{3}{4}c^2 + \frac{1}{4}(a-b)^2 \ge \frac{3}{4}c^2.$$

Moreover, $xy+yz+zx = (s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)$. Thus it si sufficient to prove that

$$s\sum(s-a)(s-b) \le \frac{9}{8}abc.$$

But, $\sum(s-a)(s-b) = r(4R+r)$, where r, R are respectively the in-radius, the circum-radius of the triangle whose sides are a, b, c, and $abc = 4Rrs$. Thus the inequality reduces to

$$r(4R+r) \le \frac{9}{2}Rr.$$

This is simply $2r \le R$. This follows from $IO^2 = R(R-2r)$, where I is the incentre and O the circumcentre.

Solution 3: If we set $x = \lambda a$, $y = \lambda b$, $z = \lambda c$, then the inequality changes to

$$(a+b+c)^2(ab+bc+ca)^2 \le 3(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2).$$

This shows that we may assume $x+y+z=1$. Let $\alpha = xy+yz+zx$. We see that

$$\begin{aligned} x^2+xy+y^2 &= (x+y)^2 - xy \\ &= (x+y)(1-z) - xy \\ &= x+y-\alpha = 1-z-\alpha. \end{aligned}$$

Thus

$$\begin{aligned} \prod(x^2+xy+y^2) &= (1-\alpha-z)(1-\alpha-x)(1-\alpha-y) \\ &= (1-\alpha)^3 - (1-\alpha)^2 + (1-\alpha)\alpha - xyz \\ &= \alpha^2 - \alpha^3 - xyz. \end{aligned}$$

Thus we need to prove that $\alpha^2 \le 3(\alpha^2 - \alpha^3 - xyz)$. This reduces to

$$3xyz \le \alpha^2(2-3\alpha).$$

However

$$3\alpha = 3(xy+yz+zx) \le (x+y+z)^2 = 1,$$

so that $2 - 3\alpha \ge 1$. Thus it suffices to prove that $3xyz \le \alpha^2$. But

$$\begin{aligned} \alpha^2 - 3xyz &= (xy+yz+zx)^2 - 3xyz(x+y+z) \\ &= \sum_{\text{cyclic}} x^2y^2 - xyz(x+y+z) \\ &= \frac{1}{2}\sum_{\text{cyclic}} (xy-yz)^2 \ge 0. \end{aligned}$$